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CURVATURE OF QUASI-SYMMETRIC SIEGEL DOMAINS

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1. Notation, definitions and basic facts

I. Satake has introduced the concept of quasi-symmetric domains. They occur as fibers in certain fiberings of symmetric domains over their boundary components, and they are contained in the larger class of spaces called homogeneous Siegel domains. The homogeneous bounded domains are biholomorphically equivalent to the homogeneous Siegel domains, and the symmetric bounded domains are equivalent to those quasi-symmetric domains that satisfy a certain additional identity, by a theorem of Satake. The quasi-symmetric domains have some convenient algebraic properties, and Satake has classified them algebraically. We work out the basic differential geometric properties of these spaces, such as Bergman metric, Bergman connection, curvature tensor, and holomorphic (bi)-sectional curvature. We also give a differential geometric proof of Satake's symmetry condition, given that the space is quasi-symmetric. The author is very indebted to his thesis adviser, Professor S. Kobayashi.

Let $\mathfrak{D}(\Omega, F) = \{(z, u) \in \mathbf{C}^n \times \mathbf{C}^m | \operatorname{Im} z - F(u, u) \in \Omega\}$ be a Siegel domain (of the second kind), defined by the cone Ω in $\mathbf{R}^n$, (Ω open, convex, not containing a whole straight line), and the Ω-hermitian form F with values in $\mathbf{C}^m$–(F is $\mathbf{C}$-linear in first variable, $\overline{F(u_1, u_2)} = F(u_2, u_1)$ and $F(u, u) \in$ (closure of Ω) $- \{0\}$ if $u \neq 0$). Identifying $\mathbf{C}^{n+m}$ with $\mathbf{C}^n \times \mathbf{C}^m$, and denoting the affine transformations of $\mathbf{C}^{n+m}$ by $Aff(\mathbf{C}^{n+m})$, we let

$$Aff(\Omega, F) := \{ g \in Aff(\mathbf{C}^{n+m}) | g\mathfrak{D}(\Omega, F) = \mathfrak{D}(\Omega, F)\},$$
$$Gl(\Omega, F) := Aff(\Omega, F) \cap Gl(n + m, \mathbf{C}).$$

We also let

$$G(\Omega) := \{A \in Gl(n, \mathbf{R}) | A\Omega = \Omega\}.$$

As is well-known [5], [7], we have

$$Aff(\Omega, F) = \{(A, \tilde{A}, a, b) \in G(\Omega) \times Gl(m, \mathbf{C}) \times \mathbf{R}^n \times \mathbf{C}^m | \\ AF(v_1, v_2) = F(\tilde{A}v_1, \tilde{A}v_2) \forall v_1, v_2 \in \mathbf{C}^m\},$$

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with action

$$(A, \tilde{A}, a, b)(z, u) = (Az + a + 2iF(\tilde{A}u, b) + iF(b, b), \tilde{A}u + b).$$

The group multiplication in $Aff(\Omega, F)$ is

(1) $(A, \tilde{A}, a, b)(B, \tilde{B}, c, d) = (AB, \tilde{A}\tilde{B}, a + Ac + 2\,\mathrm{Im}\, F(b, \tilde{A}d), b + \tilde{A}d)$,

and $(I, I, 0, 0)$ is the unit element. One calculates the Lie algebra to be

$$\mathfrak{L}\, Aff(\Omega, F)\{(X, \tilde{X}, a, b) \in \mathfrak{g}(\Omega) \times \mathfrak{gl}(m, \mathbf{C}) \times \mathbf{R}^n \times \mathbf{C}^m | XF(v_1, v_2) = F(\tilde{X}v_1, v_2) + F(v_1, \tilde{X}v_2) \forall v_1, v_2 \in \mathbf{C}^m\},$$

where $\mathfrak{g}(\Omega)$ is the Lie algebra of $G(\Omega)$. The bracket product is

$$[(X, \tilde{X}, a, b), (Y, \tilde{Y}, c, d)] = ([X, Y], [\tilde{X}, \tilde{Y}], Xc - Ya + 4\,\mathrm{Im}\, F(b, d), \tilde{X}d - \tilde{Y}b). \tag{2}$$

Now $Aff(\Omega, F) \subset \mathrm{Hol}(\Omega, F) :=$ group of holomorphic automorphisms of $\mathfrak{D}(\Omega, F)$, and if $\mathfrak{g}(\Omega, F)$ is the Lie algebra of $\mathrm{Hol}(\Omega, F)$, then we have an anti-isomorphism of $\mathfrak{g}(\Omega, F)$ with the Lie algebra of complete holomorphic vector fields on $\mathfrak{D}(\Omega, F)$. The vector field corresponding to $Z \in \mathfrak{g}(\Omega, F)$ has the value

$$Z_{(z,u)} := \frac{d}{dt}\bigg|_{t=0} \{(\exp tZ)(z, u)\} \in T_{(z,u)}\mathfrak{D}(\Omega, F)$$

at (z, u), where $T_{(z,u)}$ is the real tangent space at (z, u). More precisely, its value is the vector $\check{Z}_{(z,u)} \in \mathfrak{T}_{(z,u)}\mathfrak{D}(\Omega, F)$ such that $\check{Z}_{(z,u)} = \frac{1}{2}(Z_{(z,u)} - iJZ_{(z,u)})$, where $\mathfrak{T}\mathfrak{D}(\Omega, F)$ is the holomorphic tangent bundle and J is the complex structure. Let now $\partial_z = \partial/\partial z = (\partial/\partial z^1, \cdots, \partial/\partial z^n) = (\partial_{z^1}, \cdots, \partial_{z^n})$, and for $a \in \mathbf{R}^n$ let $a \cdot \partial_z := \Sigma\, a^i \partial_{z^i}$. Use similar notation for u. Then one calculates that

$$(X, \tilde{X}, a, b)\check{}_{(z,u)} = a \cdot \partial_z + (2iF(u, b) \cdot \partial_z + b \cdot \partial_u) + (Xz \cdot \partial_z + \tilde{X}u \cdot \partial_u). \tag{3}$$

In general, we have a grading

$$\mathfrak{g}(\Omega, F) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_\lambda$ is the λ-eigenspace for $ad(z \cdot \partial_z + \frac{1}{2}u \cdot \partial_u)$. We have

$$\mathfrak{L}\, Aff(\Omega, F) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0,$$

and $a \cdot \partial_z \in \mathfrak{g}_{-1}$, $2iF(u, b) \cdot \partial_z + b \cdot \partial_u \in \mathfrak{g}_{-1/2}$, $Xz \cdot \partial_z + \tilde{X}u \cdot \partial_u \in \mathfrak{g}_0$. From now on, let Ω be *self-dual* with respect to a positive-definite inner product $\langle\, ,\rangle$ on $\mathbf{R}^n$, in the sense that $\Omega = \Omega^* := \{t \in \mathbf{R}^n | \langle y, y' \rangle > 0\ \forall y' \in$ closure $\Omega - \{0\}\}$ *and* $G(\Omega)$ acts transitively on Ω. Then

Fact 1. [8]. $G(\Omega)$ is an open subgroup of a reductive real algebraic group and the isotropy subgroup K_a of $G(\Omega)$ at any point $a \in \Omega$ is a maximal compact subgroup. There exists an element $e \in \Omega$ such that $K_e = \{A \in G(\Omega) | A' = A^{-1}\}$, where the prime is the adjoint with respect to $\langle\, , \rangle$. The Cartan involution of $\mathfrak{g}(\Omega)$ at e is $X \mapsto -X'$, and the Cartan decomposition is therefore $\mathfrak{g}(\Omega) = \mathfrak{k}_e + \mathfrak{p}_e$ where

$$\mathfrak{k}_e = \{X \in \mathfrak{g}(\Omega) | X' = -X\} = \{X \in \mathfrak{g}(\Omega), Xe = 0\},$$

$$\mathfrak{p}_e = \{X \in \mathfrak{g}(\Omega) | X' = X\}.$$

We fix the base point $e \in \Omega$. Observing that $\mathfrak{g}(\Omega) \subset \mathfrak{gl}(n, \mathbf{R})$ consists of certain endomorphisms of $\mathbf{R}^n$, one makes the

Definition 1 [8]. It is easily seen that there is a unique element $T_a \in \mathfrak{p}_e$ such that $T_a e = a$ for any given $a \in \mathbf{R}^n$. In particular, $T_e = \mathrm{id}_{\mathbf{R}^n}$.

The mapping $\mathbf{R}^n \ni a \mapsto T_a \in \mathfrak{p}_e$ is a linear isomorphism, and one sees easily that under the isomorphism $\mathfrak{p}_e \to T_e(\Omega)$ given by $X \mapsto d/dt|_{t=0}\{(\exp tX)e\}$, we have $T_a \mapsto a \cdot \partial_y$, where $T_e(\Omega)$ is the tangent space at e, and y is the standard coordinate on $\mathbf{R}^n$.

Definition 2 [8]. Let $a_1 \circ a_2 = T_{a_1}(a_2)$ for $a_1, a_2 \in \mathbf{R}^n$. It is known [8] that under this product $\mathbf{R}^n$ becomes a (commutative) formally real Jordan algebra with unit e. We also need

$$a \circ Xe = Xa \quad \text{for } X \in \mathfrak{p}_e. \tag{4}$$

In fact $a \circ Xe = T_a Xe = [T_a, X]e + XT_a e = Xa$, since $[T_a, X] \in \mathfrak{k}_e$.

Definition 3 [8]. Given a Siegel domain $\mathfrak{D}(\Omega, F)$, we say that $\tilde{A} \in \mathfrak{gl}(m, \mathbf{C})$ is *associated* to $A \in \mathfrak{g}(\Omega)$ if

$$AF(v_1, v_2) = F(\tilde{A}v_1, v_2) + F(v_1, \tilde{A}v_2) \ \forall v_1, v_2 \in \mathbf{C}^m. \tag{5}$$

Definition 4 [8]. Extending $\langle\, , \rangle$ to a $\mathbf{C}$-bilinear symmetric form on $\mathbf{C}^n \times \mathbf{C}^n$, we put, for $a \in \mathbf{R}^n$,

$$F_a(v_1, v_2) = \langle a, F(v_1, v_2)\rangle.$$

We have that F_a is a hermitian form on $\mathbf{C}^m$, and that it is positive-definite if $a \in \Omega^*$, by virtue of the definition of Ω^*. So if Ω is self-dual, then F_e is a positive-definite hermitian form on $\mathbf{C}^m$.

Definition 5 [8]. If Ω is self-dual, for $a \in \mathbf{R}^n$ let $R_a \in \mathfrak{gl}(m, \mathbf{C})$ be given by

$$F_a(v_1, v_2) = 2F_e(v_1, R_a v_2),$$

i.e.,

$$\langle a, F(v_1, v_2)\rangle = 2\langle e, F(v_1, R_a v_2)\rangle.$$

If $\mathcal{H}(F_e)$ are the F_e-selfadjoint transformations of $\mathbf{C}^m$, and $\mathcal{P}(F_e)$ is the set (cone) of the positive definite subsets of $\mathcal{H}(F_e)$, then $R_a \in \mathcal{H}(F_e)$, and $R_a \in \mathcal{P}(F_e)$ for $a \in \Omega$.

Remark. Satake uses an F which is conjugate to ours, but this does not affect the definition of R_a.

We also let R denote the map $\mathbf{R}^n \ni a \mapsto R_a \in \mathcal{H}(F_e)$ and also the $\mathbf{C}$-linear extension $\mathbf{C}^n \mapsto \mathfrak{gl}(m, \mathbf{C})$ of this, [8]. The relation (5) can be written, [8],

$$R_{A'a} = \tilde{A}^* R_a + R_a \tilde{A} \quad \text{for } a \in \mathbf{R}^n. \tag{6}$$

From [8] we quote

Fact 2. If $\mathfrak{D}(\Omega, F)$ is a Siegel domain with Ω self-dual, then the following conditions are equivalent:

(i) For every $a \in \mathbf{R}^n$, R_a is associated to T_a.

(ii) The map $R: a \mapsto R_a$ of $\mathbf{R}^n$ into $\mathcal{H}(F_e)$ satisfies

$$R_{a_1 \circ a_2} = R_{a_1} R_{a_2} + R_{a_2} R_{a_1}.$$

(iii) There exists a (unique) Lie algebra homomorphism β: $\mathfrak{g}(\Omega) \to \mathfrak{gl}^0(m, \mathbf{C}) := \{X \in \mathfrak{gl}(m, \mathbf{C}) | \text{trace } X \in \mathbf{R}\}$ such that

$$\beta(X) \text{ is associated to } X, \tag{7}$$

i.e.,

$$R_{X'a} = \beta(X)^* R_a + R_a \beta(X) \ \forall a \in \mathbf{R}^n,$$
$$\beta(X') = \beta(X)^*. \tag{8}$$

(iv) The projection map $\mathfrak{g}_0 \ni (X, \tilde{X}) \mapsto X \in \mathfrak{g}(\Omega)$ is surjective.

($\mathfrak{g}_0$ is a term in the decomposition

$$\mathfrak{g}(\Omega, F) = \mathfrak{g}_{-1} + \mathfrak{g}_{-1/2} + \mathfrak{g}_0 + \mathfrak{g}_{1/2} + \mathfrak{g}_1.)$$

Now finally we can define the spaces which we want to study.

Definition 6 [8]. A Siegel domain $\mathfrak{D}(\Omega, F)$ with self-dual Ω is said to be *quasi-symmetric* if the equivalent conditions in Fact 2 are satisfied.

A quasi-symmetric domain is homogeneous, since Ω and therefore also $\mathfrak{D}(\Omega, F)$ are homogeneous [5].

To have the situation as simply as possible, we have the

Definition 7. A cone $\Omega \subset \mathbf{R}^n$ is said to be *decomposable* if there exist nonzero linear subspaces U_1, U_2 of $\mathbf{R}^n$, and cones $\Omega_1 \subset U_1$, $\Omega_2 \subset U_2$ such that $\mathbf{R}^n = U_1 \oplus U_2$ and $\Omega = \Omega_1 \times \Omega_2$. If no such decomposition exists, the cone is said to be *indecomposable*.

Similarly we have

Definition 8. A complex manifold biholomorphic to a homogeneous bounded domain is said to be *decomposable* if it is biholomorphic to the product of two nontrivial homogeneous bounded domains. If no such decomposition exists, the manifold is said to be *indecomposable*.

It has been shown [2] that any homogeneous bounded domain is biholomorphic to the product of indecomposable homogeneous bounded domains, and also that a homogeneous Siegel domain $\mathfrak{D}(\Omega, F)$ is indecomposable if and only if Ω is indecomposable. (See also [10].)

Because of the above, we restrict attention to (homogeneous) Siegel domains $\mathfrak{D}(\Omega, F)$ with a self-dual and indecomposable cone Ω satisfying the condition of quasi-symmetry.

Up to isomorphism the self-dual indecomposable cones can be described as follows [10], [11].

I. Let $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$, the sets of real numbers, complex numbers, and quaternions respectively, and for each integer $m \geqslant 1$, let

$$\mathcal{H}_m(\mathbf{F}) = \{X \in \mathbf{M}_m(\mathbf{F}) | X^* = X\},$$

where $\mathbf{M}_m(\mathbf{F})$ is the set of $m \times m$ matrices with coefficients in $\mathbf{F}$, and $X^* = \bar{X}'$ is the conjugate transpose, using the standard conjugation on $\mathbf{F}$. Then the set $\mathcal{P}_m(\mathbf{F}) = \{X \in \mathcal{H}_m(\mathbf{F}) | X \text{ positive-definite}\}$ is an indecomposable cone which is self-dual with respect to the inner product

$$\langle X, Y \rangle = \operatorname{trace}(XY)$$

on the real vector space $\mathcal{H}_m(\mathbf{F})$. We call these cones *classical cones*. The set $\Delta_m(\mathbf{F})$ of upper triangular matrices in $\mathbf{M}_m(\mathbf{F})$ with real positive diagonal entries acts simply transitively on $\mathcal{P}_m(\mathbf{F})$ by

$$(t, X) \mapsto tXt^* \quad \text{for } X \in \mathcal{P}_m(\mathbf{F}) \text{ and } t \in \Delta_m(\mathbf{F}).$$

II. For $n \geqslant 3$ we define the quadratic form Q_n on $\mathbf{R}^n$ by

$$Q_n(x) = x_1 x_2 - x_3^2 - \cdots - x_n^2.$$

We put $S_n = \{x \in \mathbf{R}^n | Q_n(x) > 0,\ x_1 > 0\}$. Then S_n is an indecomposable cone which is self-dual with respect to the ordinary inner product on $\mathbf{R}^n$. We call these cones *spherical cones*. The connected component of the identity of the group of similitudes of Q_n acts transitively on S_n. (We modify the inner product slightly in §2.)

III. There is also an *exceptional* cone $\mathcal{P}_3$ (Cayley) which we exclude here, since Satake has proved that a quasi-symmetric domain with this cone must be the tube domain defined by it, and we are mainly interested in Siegel domains of the second kind. (Reason for the exclusion is simply that this case, being symmetric, is already well understood.) So we *agree* to forget about this cone in all statements belows.

The key fact we need in order to establish a connection between the differential geometry of $\mathfrak{D}(\Omega, F)$ and Satake's algebraic description is

Fact 3 [5]. Let $\mathfrak{D}(\Omega, F)$ be a homogeneous Siegel domain. The Bergman kernel function is of the form $\mathcal{K} = \lambda \circ \Phi$, where λ is a positive function on Ω, and Φ is the map

$$\Phi(z, u) = \operatorname{Im} z - F(u, u)$$

of $\mathfrak{D}(\Omega, F)$ onto Ω. Moreover, if $(A, \tilde{A}) \in Gl(\Omega, F)$, then

$$\lambda(Ax) = |\det A|^{-2} |\det \tilde{A}|^{-2} \lambda(x)$$

for $x \in \Omega$.

Observe that the Bergman metric is defined, since $\mathfrak{D}(\Omega, F)$ is biholomorphic to a bounded domain, [5], and hence we can transfer the metric from that domain just as in the case of the upper half-plane.

2. The Bergman metric

We need some lemmas. Recall (§1) that $\Delta_p(\mathbf{F})$ denotes the group of upper triangular matrices in $\mathbf{M}_p(\mathbf{F})$ with positive entries on the diagonal, where $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$. The image of $A \in \Delta_p(\mathbf{F})$ under the mapping $\Delta_p(\mathbf{F}) \to G(\mathfrak{P}_p(\mathbf{F}))$ is denoted here by $\check{A}$. We have $\check{A}Y = AYA^*$ for $Y \in \mathfrak{P}_p(\mathbf{F})$. Now $\check{A} \in Gl(\mathcal{H}_p(\mathbf{F}))$, and $\mathcal{H}_p(\mathbf{F})$ is a real vector space of dimension $d = \frac{1}{2}p(p+1), p^2, 2p^2 - p$ for $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ respectively.

If

$$A = \begin{bmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_p \end{bmatrix},$$

let $\det A = a_1 \cdots a_p$ also in the quaternionic case. We have

Lemma 1. $\det \check{A} = (\det A)^\varepsilon$ *for* $A \in \Delta_p(\mathbf{F})$, *where* $\varepsilon = p + 1, 2p, 4p - 2$ *for* $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ *respectively*.

Proof. If there is such an ε, we find it by replacing A by sA with $s > 0$. Then $(sA)^\vee = s^2\check{A}$, and $\det s^2\check{A} = s^{2d} \det \check{A} = s^{2d}(\det A)^\varepsilon$. On the other hand, $(\det sA)^\varepsilon = (s^p \det A)^\varepsilon = s^{p\varepsilon}(\det A)^\varepsilon$. So $\varepsilon = 2d/p$.

We have only to prove that $\det A = 1 \Rightarrow \det \check{A} = 1$. Using the Lie algebra, we have to show that if

$$X = \begin{bmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_p \end{bmatrix} \in \mathfrak{L}\Delta_p(\mathbf{F})$$

has trace zero, then so has the endomorphism

$$Y \mapsto XY + YX^*$$

of $\mathcal{H}_p(\mathbf{F})$. Using a standard basis for $\mathcal{H}_p(\mathbf{F})$, we see that this is an elementary computation, which is omitted here. q.e.d.

We use $z = x + iy$ as (part of) coordinates on $\mathfrak{D}(\Omega, F)$. In order not to have any confusion, we use a different name t for coordinates on $\Omega \subset \mathbf{R}^n$.

Also observe that we can take $e = \mathrm{id} \in \mathcal{P}_p(\mathbf{F})$ as the base point satisfying the conditions in §1 with respect to the metric introduced in that section.

Lemma 2. *There is a C^∞ solution $\check{A}(t) \in G(\mathcal{P}_p(\mathbf{F}))$ of the equation $t = \check{A} \cdot e$ for t near $e = \mathrm{id} \in \mathcal{P}_p(\mathbf{F})$, satisfying the condition*: $(\det \check{A}(t))^2$ *is a homogeneous polynomial of some degree l in $t \in \mathbf{R}^d$. (The basis for $\mathcal{H}_p(\mathbf{F}) \approx \mathbf{R}^d$ is inessential.)*

Proof. Consider first the cases $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$. By §1, there is $A \in \Delta_p(\mathbf{F})$ for given $t \in \mathcal{P}_p(\mathbf{F})$ such that $t = AA^* = \check{A} \cdot e$. By Lemma 1 we have $(\det t)^\varepsilon = (\det A \cdot \det A^*)^\varepsilon = (\det A)^{2\varepsilon} = (\det \check{A})^2$, since A is triangular and has real diagonal entries. The degree of the homogeneous polynomial $(\det t)^\varepsilon$ is p_ε. Since (§1) $\Delta_p(\mathbf{F})$ is simply transitive on $\mathcal{P}_p(\mathbf{F})$, the rest is clear.

A similar computation works in the quaternionic case. Here we have to use Dieudonné's theory of noncommutative determinants, as can be found in [1, Chapter IV]. The determinants now take values in the semigroup obtained by adding 0 to the abelian group $\mathbf{H}^*/[\mathbf{H}^*, \mathbf{H}^*]$, where $\mathbf{H}^*$ is the multiplicative group of nonzero quaternions, and $[\mathbf{H}^*, \mathbf{H}^*]$ is the commutator subgroup. The computation of a determinant in this semigroup is formally the same as in the ordinary case, and we can proceed as before. q.e.d.

We need these lemmas also for the spherical cone S_n. Since the proofs are analogous to the above ones, we sketch them.

First we write $t = (t_1, t_2, \cdots, t_n)$ as a symmetric "matrix":

$$t = \begin{bmatrix} t_1 & \tilde{t} \\ \tilde{t} & t_2 \end{bmatrix},$$

where $\tilde{t} = (t_3, \cdots, t_n) \in \mathbf{R}^{n-2}$. The form $Q(t)$ is like a determinant:

$$Q(t) = t_1 t_2 - t_3^2 - \cdots - t_n^2 = t_1 t_2 - \tilde{t}^2 =: \det t.$$

We let $\Delta = \{\left(\begin{smallmatrix} a & \tilde{v} \\ 0 & b \end{smallmatrix}\right) | a > 0,\ b > 0,\ \tilde{v} \in \mathbf{R}^{n-2}\}$ be the upper triangular group (with positive diagonal elements), with usual group operations:

$$\begin{pmatrix} a & \tilde{v} \\ 0 & b \end{pmatrix}\begin{pmatrix} c & \tilde{w} \\ 0 & d \end{pmatrix} = \begin{pmatrix} ac & a\tilde{w} + d\tilde{v} \\ 0 & bd \end{pmatrix}, \begin{pmatrix} a & \tilde{v} \\ 0 & b \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & -a^{-1}b^{-1}\tilde{v} \\ 0 & b^{-1} \end{pmatrix}.$$

If $\tilde{t} \in \mathbf{R}^{n-2}$ and $r \in \mathbf{R}$, then Δ acts to the left on $\left(\begin{smallmatrix} \tilde{t} \\ r \end{smallmatrix}\right)$-vectors and $\left(\begin{smallmatrix} r \\ \tilde{t} \end{smallmatrix}\right)$-vectors by $\left(\begin{smallmatrix} a & \tilde{v} \\ 0 & b \end{smallmatrix}\right)\left(\begin{smallmatrix} \tilde{t} \\ r \end{smallmatrix}\right) = \left(\begin{smallmatrix} a\tilde{t} + r\tilde{v} \\ br \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a & \tilde{v} \\ 0 & b \end{smallmatrix}\right)\left(\begin{smallmatrix} r \\ \tilde{t} \end{smallmatrix}\right) = \left(\begin{smallmatrix} a\tilde{t} + \tilde{v} \cdot \tilde{t} \\ b\tilde{t} \end{smallmatrix}\right)$. Similarly the lower triangular group Δ' acts to the right on $(\tilde{t}, r)$– and $(r, \tilde{t})$-vectors, and one checks that products of the form

$$\begin{pmatrix} a & \tilde{v} \\ 0 & b \end{pmatrix}\begin{bmatrix} t_1 & \tilde{t} \\ \tilde{t} & t_2 \end{bmatrix}\begin{pmatrix} a & 0 \\ \tilde{v} & b \end{pmatrix} = \begin{bmatrix} a^2 t_1 + 2a\tilde{v} \cdot \tilde{t} + t_2 \tilde{v}^2 & ab\tilde{t} + bt_2\tilde{v} \\ ab\tilde{t} + bt_2\tilde{v} & b^2 t_2 \end{bmatrix}$$

are well-defined elements of S_n for $\begin{pmatrix} t_1 & \tilde{t} \\ \tilde{t} & t_2 \end{pmatrix} \in S_n$, with determinant $a^2b^2Q(t) > 0$ and positive diagonal elements. (Enough to see that $b^2t_2 > 0$.)

In this way we have a homomorphism

$$h: \Delta \ni A \mapsto \check{A} \in G(S_n),$$

as $\check{A} \cdot t = AtA'$. Then Δ is transitive on S_n, for the element

$$A = \frac{1}{\sqrt{t_2}} \begin{pmatrix} \sqrt{Q(t)} & \tilde{t} \\ 0 & t_2 \end{pmatrix}$$

sends $e := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in S_n$ to t.

The stability group is trivial, and we note that

(1) $$(\det A)^2 = Q(t) = \det t.$$

Furthermore, we have

(2) $$\det \check{A} = (\det A)^n.$$

To see that we replace A by sA with $s > 0$, as before. Then $\det(sA)^\vee = \det(s^2\check{A}) = s^{2n} \det \check{A}$, and on the other hand $(\det sA)^n = (s^2 \det A)^n = s^{2n}(\det A)^n$. So we have only to check that $\det \check{A} = 1$ if $\det A = 1$. We compute that the Lie algebra of $\{A \in \Delta | \det A = 1\}$ is $\{\begin{pmatrix} a & \tilde{v} \\ 0 & -a \end{pmatrix}\}$ with bracket

$$\left[\begin{pmatrix} a & \tilde{v} \\ 0 & -a \end{pmatrix}, \begin{pmatrix} b & \tilde{w} \\ 0 & -b \end{pmatrix}\right] = \begin{pmatrix} 0 & 2a\tilde{w} - 2b\tilde{v} \\ 0 & 0 \end{pmatrix},$$

and that

$$\exp s\begin{pmatrix} a & \tilde{v} \\ 0 & -a \end{pmatrix} = \sum_{j=0}^{\infty} \frac{s^j}{j!} \begin{pmatrix} a & \tilde{v} \\ 0 & -a \end{pmatrix}^j.$$

Doing the same for Δ' and differentiating the equation

$$h\left(\exp s\begin{pmatrix} a & \tilde{v} \\ 0 & -a \end{pmatrix}\right) \cdot t = \left(\exp s\begin{pmatrix} a & \tilde{v} \\ 0 & -a \end{pmatrix}\right) t \left(\exp s\begin{pmatrix} a & 0 \\ \tilde{v} & -a \end{pmatrix}\right)$$

with respect to s, we find

$$h\begin{pmatrix} a & \tilde{v} \\ 0 & -a \end{pmatrix}: t \mapsto \begin{pmatrix} a & \tilde{v} \\ 0 & -a \end{pmatrix} t + t\begin{pmatrix} a & 0 \\ \tilde{v} & -a \end{pmatrix}.$$

Putting $L := h\begin{pmatrix} a & \tilde{v} \\ 0 & -a \end{pmatrix}$ and using the basis

$$u_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad u_j = \begin{pmatrix} 0 & e_j \\ e_j & 0 \end{pmatrix}, \quad j = 3, \cdots, n$$

for the space of symmetric "matrices" $\{\begin{pmatrix} t_1 & \tilde{t} \\ \tilde{t} & t_2 \end{pmatrix}\}$, where $\{e_j\}$ is the standard basis for $\mathbf{R}^{n-2} = \{(t_3, \cdots, t_n)\}$, we find $Lu_1 = 2au_1$, $Lu_2 = -2au_2 + \sum_{j=3}^n v_j u_j$, $Lu_j = 2c_j u_1$, $j = 3, \cdots, n$, and hence trace $L = 0$. So we get the lemmas as before:

Lemma 1′. $\det \check{A} = (\det A)^n$ *for* $A = \begin{pmatrix} a & \tilde{v} \\ 0 & b \end{pmatrix} \in \Delta = \Delta_n$.

Lemma 2′. *There is a C^∞ solution $\check{A}(t) \in G(S_n)$ of the equation $t = \check{A} \cdot e$ for t near $e = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right) = (1, 1, 0, \cdots, 0) \in S_n$, satisfying the condition*: $(\det \check{A}(t))^2$ *is a homogeneous polynomial* ($(\det t)^n$) *of degree* $l (= 2n)$ *in* $t \in \mathbf{R}^n$.

Now by Fact 1 we have $\mathfrak{g}(\Omega) = \mathfrak{D}\mathfrak{g}(\Omega) \oplus \mathfrak{Z}\mathfrak{g}(\Omega)$, where $\mathfrak{D}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, and $\mathfrak{Z}\mathfrak{g}$ is the center of $\mathfrak{g}$ and $\Omega = \mathcal{P}_p(\mathbf{F})$ or S_n. We have $\dim \mathfrak{Z}\mathfrak{g} = 1$ since Ω is indecomposable, (see [9]), and we let Z_0 be a generator for $\mathfrak{Z}\mathfrak{g}$. Then any element $X \in \mathfrak{g}(\Omega)$ can be written $X = Y + cZ_0$ with $Y \in \mathfrak{D}\mathfrak{g}$ and $c \in \mathbf{R}$, and hence $\operatorname{trace} X = c \operatorname{trace} Z_0$. This implies $\det \exp X = \exp \operatorname{trace} X = \exp c \operatorname{trace} Z_0$. We cannot have $\operatorname{trace} Z_0 = 0$, since then the determinant of any element in $G(\Omega)^0$, the identity component of $G(\Omega)$, would be 1, in contradiction to Lemmas 1 and 1′.

Consider now the homomorphism $\beta: \mathfrak{g}(\Omega) \to \mathfrak{g}l^0(m, \mathbf{C}) \subset \mathfrak{g}l(m, \mathbf{C})$ given in Fact 2. We have $\beta X = \beta Y + c\beta Z_0$ for the above X, and here $\beta Y \in \mathfrak{D}\mathfrak{g}l(m, \mathbf{C})$. Therefore $\operatorname{trace} \beta X = c \operatorname{trace} \beta Z_0$, which gives

$$\det \beta \exp X = (\exp c \operatorname{trace} Z_0)^{\operatorname{trace} \beta Z_0/\operatorname{trace} Z_0}$$
$$= (\det \exp X)^r \text{ for all } X \in \mathfrak{g}(\Omega),$$

where $r = \operatorname{trace} \beta Z_0/\operatorname{trace} Z_0$, by observng that β extends to a group homomorphism $G(\Omega)^0 \to Gl(m, \mathbf{C})$, [8], [9]. We thus have

Lemma 3. $\det \beta\check{A} = (\det \check{A})^r$ *for any* $\check{A} \in G(\Omega)^0$, *the identity component of* $G(\Omega)$, *where* $r \in \mathbf{R}$ *is independent of* $\check{A}$.

Here $(\check{A}, \beta\check{A}) \in Gl(\Omega, F)$. (We still write $\check{A}$ in order not to confuse with elements of $\Delta_p(\mathbf{F})$ or Δ.)

Using Lemmas 2 and 2′ and the notation there, and combining with Lemma 3, we have

Lemma 4. $(\det \beta A(t))^2$ *is a homogeneous function of degree lr in t.*

We now turn to the Bergman metric of a quasi-symmetric Siegel domain $\mathfrak{D}(\Omega, F)$ with $\Omega \subset \mathbf{R}^n$ as above. Putting $Z^{n+k} := u^k$, $k = 1, \cdots, m$ for the moment, where $F: \mathbf{C}^m \times \mathbf{C}^m \to \mathbf{C}^n$, we have

$$ds^2 = ds^2_{\mathfrak{D}(\Omega, F)} = 2 \sum_{i,j=1}^{n+m} \frac{\partial^2 \log \mathcal{K}}{\partial z^i \partial \bar{z}^j} dz^i d\bar{z}^j =: 2 \sum_{i,j=1}^{n+m} g_{i\bar{j}}\, dz^i\, d\bar{z}^j,$$

where $\mathcal{K} = \lambda \circ \Phi$ is as described in Fact 3, and so $\lambda(t) = \mathcal{K}(it, 0) > 0$. (We also write $\langle\, ,\, \rangle_p$ for this metrical product at the point p later on.) By Fact 3 and Lemma 3 we have for $\check{A} \in G(\Omega)^0$ that

$$\lambda(\check{A}t) = |\det \check{A}|^{-2} |\det \beta\check{A}|^{-2} \lambda(t) = |\det \check{A}|^{-2(1+r)} \lambda(t).$$

Hence $w(\check{A}t) = |\det \check{A}|^{-1} w(t)$, where $w := \lambda^{1/(2+2r)}$, and so $w\, dt^1 \wedge \cdots \wedge dt^n$ is a $G(\Omega)^0$-invariant volume form on Ω. (We cannot have $1 + r = 0$, since then λ and hence $\mathcal{K}$ would be constant. But $\mathcal{K}$ cannot be constant since $\mathfrak{D}(\Omega, F)$ is (equivalent to) a bounded domain.)

The following is clear.

Lemma 5. *For the* $G(\Omega)^0$*-invariant Riemannian metric*

$$ds^2_\Omega = \sum_{i,j=1}^{n} \frac{\partial^2 \log w}{\partial t^i \partial t^j}\, dt^i\, dt^j \text{ on } \Omega,$$

we have $ds^2_\Omega = ds^2_{\mathfrak{D}(\Omega)|i\Omega}$, *where* $ds^2_{\mathfrak{D}(\Omega)}$ *is the Bergman metric on the tube domain* $\mathfrak{D}(\Omega) = \mathbf{R}^n + i\Omega$.

With obvious indexing, slightly different from the above, and using the summation convention, we get from $\log \mathcal{K} = (\log \lambda) \circ \Phi$ and $t^j = \Phi^j(z, u) = \operatorname{Im} z^j - F^j_{\alpha\bar{\beta}} u^\alpha \bar{u}^\beta$ that

$$\frac{\partial^2 \log \mathcal{K}}{\partial u^\alpha \partial \bar{u}^\beta} = \frac{\partial^2 \log \lambda}{\partial t^i \partial t^j}(\Phi) \cdot \frac{\partial \Phi^i}{\partial u^\alpha} \cdot \frac{\partial \Phi^j}{\partial \bar{u}^\beta} + \frac{\partial \log \lambda}{\partial t^j}(\Phi) \cdot \frac{\partial^2 \Phi^j}{\partial u^\alpha \partial \bar{u}^\beta}.$$

Now $\partial \Phi^i / \partial u^\alpha = -F^i{}_{\alpha\bar{\beta}} \bar{u}^\beta$ and $\partial^2 \Phi^j / \partial u^\alpha \partial \bar{u}^\beta = -F^j{}_{\alpha\bar{\beta}}$, so at $o = (ie, 0)$, which we choose as base point in $\mathfrak{D}(\Omega, F)$,

$$(3) \qquad g_{\alpha\bar{\beta}0} = -F^j_{\alpha\bar{\beta}} \frac{\partial \log \lambda}{\partial t^j}(e); \quad \alpha, \beta = 1, \cdots, m.$$

Similarly we have $\partial^2 \log \mathcal{K} / \partial z^i \partial \bar{u}^\beta = 0$ at o, since $\partial \Phi^j / \partial \bar{u}^\beta = -F^j_{\alpha\bar{\beta}} u^\alpha$ and $\partial^2 \Phi^j / \partial z^i \partial \bar{u}^\beta \equiv 0$. So

$$(4) \qquad g_{i\bar{\beta}_o} = 0; \quad i = 1, \cdots, n; \beta = 1, \cdots, m.$$

Further

$$(5) \qquad g_{i\bar{j}} = \frac{1}{4} \frac{\partial^2 \log \lambda}{\partial t^i \partial t^j} \circ \Phi,$$

since $\partial \Phi^k / \partial z^i = -\frac{1}{2}\sqrt{-1}\, \delta_i^k$, where δ_i^k is the Kronecker symbol. This gives , at any point of $\mathfrak{D}(\Omega, F)$,

$$(6) \qquad g(\partial_{x^i}, \partial_{y^j}) = 2 \operatorname{Im} g_{i\bar{j}} = 0.$$

Definition 1. For any point $p \in \mathfrak{D}(\Omega, f)$, we let $\mathcal{V}_p \subset T_p\mathfrak{D}(\Omega, F)$ denote the *vertical space* at p, i.e., the tangent space to the fiber of $\Phi: \mathfrak{D}(\Omega, F) \to \Omega$ through p. Similarly we let $\mathcal{H}_p \subset T_p\mathfrak{D}(\Omega, F)$ denote the *horizontal space* at p, i.e., the orthogonal complement to $\mathcal{V}_p$ with respect to ds^2.

Looking at $t^j = \Phi^j(z, u) = y^j - F^j(u, u)$ and using (4), (5), (6), we get

Lemma 6. $\mathcal{V}_0 = \{a \cdot \partial_x\} \oplus \{b \cdot \partial_u + \bar{b} \cdot \partial_{\bar{u}}\}$ *and* $\mathcal{H}_0 = \{a \cdot \partial_y\}$, *where* $\oplus$ *is the orthogonal sum and* $a \in \mathbf{R}^n$, $b \in \mathbf{C}^m$.

Letting π: $Aff(\Omega, F) \to G(\Omega)$ be the homomorphism $\pi(\check{A}, \tilde{A}, a, b) \mapsto \check{A}$, we easily have

Lemma 7. *The mapping* Φ: $\mathfrak{D}(\Omega, F) \to \Omega$ *is* π*-equivariant, i.e.,* $\Phi(gp) = \pi(g)\Phi(p)$ *for* $g \in Aff(\Omega, F)$ *and* $p \in \mathfrak{D}(\Omega, F)$.

Therefore the distributions $\{\mathcal{H}_p\}_{p\in\mathfrak{D}(\Omega,F)}$ and $\{\mathcal{V}_p\}_{p\in\mathfrak{D}(\Omega,F)}$ are $Aff(\Omega, F)$-invariant, and we have

Lemma 8. *$\mathcal{V}_{(z,u)} = \{a\cdot\partial_x\} + \{b\cdot\partial_u + F(b,u)\cdot\partial_y + \text{conj}\}$ and $\mathcal{H}_{(z,u)} = \{a\cdot\partial_y\}$, where $a\in\mathbf{R}^n$, $b\in\mathbf{C}^m$. Also the summands in $\mathcal{V}$ are orthogonal if $u=0$.*

Proof. Assume first that $(z,u) = (it, 0)$, and choose $g = (\hat{A}, \tilde{A}) \in Gl(\Omega, F)$ such that $\check{A}t = e$. Then $g\colon (z,u)\mapsto(\check{A}z, \tilde{A}u)$, whence (summation convention) $g_*\partial_{z^i} = \check{A}_{ji}\partial_{z^j}$, $g_*\partial_{u^\alpha} = \tilde{A}_{\beta\alpha}\partial_{u^\beta}$, or $g_*(a\cdot\partial_z) = (\check{A}a)\cdot\partial_z$, $g_*(b\cdot\partial_u) = (\tilde{A}b)\cdot\partial_u$. Since $g(it,0) = o$, we see $\mathcal{V}_{(it,0)} = \{a\cdot\partial_x\}\oplus\{b\cdot\partial_u + \bar{b}\cdot\partial_{\bar{u}}\}$ and $\mathcal{H}_{(it,0)} = \{a\cdot\partial_y\}$, by Lemma 6.

Now let (z_0, u_0) be any point, and observe that

$$g(z_0, u_0) = (i\Phi(z_0, u_0), 0),$$

where now $g = (I, I, -\text{Re}\, z_0, -u_0) \in Aff(\Omega, F)$, and that

$$g(z,u) = (z - \text{Re}\, z_0 - 2iF(u, u_0) + iF(u_0, u_0), u - u_0).$$

Then $g_*(a\cdot\partial_z) = a\cdot\partial_z$ and $g_*(b\cdot\partial_u) = -2iF(b, u_0)\cdot\partial_z + b\cdot\partial_u$, and hence also $g_*(b\cdot\partial_u + F(b,u_0)\cdot\partial_y) = -iF(b,u_0)\cdot\partial_x + b\cdot\partial_u$. The rest then follows from the first part. q.e.d.

Since $\Phi_*(a\cdot\partial_y) = a\cdot\partial_t$, and

$$\langle\partial_{y^i}, \partial_{y^j}\rangle_p = 2\,\text{Re}\, g_{i\bar{j}p} = \frac{1}{2}\frac{\partial^2\log\lambda}{\partial t^i\partial t^j}\circ\Phi(p)$$

by (5), and since $w = \lambda^{1/(2+2r)}$, Lemmas 5 and 8 give

Corollary 1. *With r as in Lemma 3, we have*

$$(1+r)ds^2_\Omega(\Phi_*Y, \Phi_*Y) = ds^2_{\mathfrak{D}(\Omega,F)}(Y, Y)$$

for any $Y\in\mathcal{H}_{(z,u)}$, and so the mapping

$$\Phi\colon \mathfrak{D}(\Omega, F)\to\Omega$$

is a Riemannian submersion [6] *when we give Ω the metric $(1+r)ds^2_\Omega$.*

Remark. We see that $1 + r > 0$.

We now have to connect the metric with the algebra in [8]. First we shall identify the inner product in $\mathcal{H}_0$ with the given inner product $\langle\ ,\ \rangle$ on $\mathbf{R}^n$. By Corollary 1 this means that we must identify $g_{\Omega,e}$ with $\langle\ ,\ \rangle$. As in Lemmas 2 and 2′, we write $t = \check{A}(t)\cdot e = A(t)eA(t)^*$ for t near $e\in\Omega$, where $\check{A}(t)$ comes from an element $A(t)\in\Delta_p(\mathbf{F})$ or Δ_n, according as Ω is classical or spherical. For

$$ds^2_\Omega = \sum\frac{\partial^2\log w}{\partial t^i\partial t^j}\,dt^i\,dt^j$$

we have $w(t) = w(\check{A}(t) \cdot e) = (\det \check{A}(t))^{-1} w(e) = (\det A(t))^{-\epsilon} w(e)$, by Lemmas 1 and 1′. We saw further in the proofs of Lemmas 2 and 2′ that $\det t = (\det A)^2$, and therefore

$$\log w(t) = -\frac{\varepsilon}{2} \log \det t + \log w(e).$$

Thus

$$ds^2_\Omega = \frac{\varepsilon}{2} \sum \left\{ \frac{1}{(\det t)^2} \cdot \frac{\partial \det t}{\partial t^i} \cdot \frac{\partial \det t}{\partial t^j} - \frac{1}{\det t} \cdot \frac{\partial^2 \det t}{\partial t^i \partial t^j} \right\} dt^i\, dt^j.$$

Since $\det e = 1$, we see

$$ds^2_{\Omega,e} = \frac{\varepsilon}{2} \sum \left\{ \frac{\partial \det t}{\partial t^i} \cdot \frac{\partial \det t}{\partial t^j} - \frac{\partial^2 \det t}{\partial t^i \partial t^j} \right\}_e dt^i\, dt^j.$$

Consider first the classical cones $\mathcal{P}_p(\mathbf{F})$, and change the indexing so that for instance for $\mathcal{P}_2(\mathbf{C})$ we have

$$t = \begin{pmatrix} t_{11} & t'_{12} + it''_{12} \\ t'_{12} - it''_{12} & t_{22} \end{pmatrix}.$$

Then $\det t = t_{11}t_{22} - (t'_{12})^2 - (t''_{12})^2$, and one verifies that

$$\langle \partial_{t_{11}}, \partial_{t_{11}} \rangle_{\Omega,e} = 1, \quad \langle \partial_{t_{11}}, \partial_{t'_{12}} \rangle_{\Omega,e} = 0, \quad \langle \partial_{t_{11}}, \partial_{t''_{12}} \rangle_{\Omega,e} = 0,$$
$$\langle \partial_{t_{11}}, \partial_{t_{22}} \rangle_{\Omega,e} = 0, \quad \langle \partial_{t'_{12}}, \partial_{t'_{12}} \rangle_{\Omega,e} = 2, \quad \langle \partial_{t'_{12}}, \partial_{t''_{12}} \rangle_{\Omega,e} = 0$$

etc., except for the factor $\varepsilon/2$. This works in the other cases too, and we have (except for $\varepsilon/2$): The $\partial_{t_{ij}}$'s are orthogonal to each other, those on the diagonal have length 1, the others have length $\sqrt{2}$. On the other hand, if

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad E'_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E''_{12} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

form a (real) basis for $\mathcal{H}_2(\mathbf{C})$, then $\langle E_{11}, E_{11} \rangle = \operatorname{trace}(E_{11}^2) = 1$, and $\langle E_{11}, E'_{12} \rangle = 0$, $\langle E_{11}, E''_{12} \rangle = 0$, $\langle E_{11}, E_{22} \rangle = 0$, $\langle E'_{12}, E'_{12} \rangle = 2$, $\langle E'_{12}, E''_{12} \rangle = 0$, etc., and again this holds in general. So we have

$$\langle a_1 \cdot \partial_t, a_2 \cdot \partial_t \rangle_{\Omega,e} = \frac{\varepsilon}{2} \langle a_1, a_2 \rangle \tag{7}$$

for $a_1, a_2 \in \mathbf{R}^n$ (= space in which Ω lies).

Consider then the spherical cone S_n. As quoted in §1, the reference [10] uses the ordinary inner product on $\mathbf{R}^n$ for this cone. Since we treat S_n as a set of symmetric "matrices", we change the product slightly: If

$$X = \begin{pmatrix} x_1 & \tilde{x} \\ \tilde{x} & x_2 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & \tilde{y} \\ \tilde{y} & y_2 \end{pmatrix} \in \mathbf{R}^n,$$

then

$$XY = \begin{pmatrix} x_1 y_1 + \tilde{x} \cdot \tilde{y} & x_1 \tilde{y} + y_2 \tilde{x} \\ y_1 \tilde{x} + x_2 \tilde{y} & x_2 y_2 + \tilde{x} \cdot \tilde{y} \end{pmatrix}$$

is well-defined, and has "trace" $x_1y_1 + x_2y_2 + 2\tilde{x}\cdot\tilde{y}$. So we define $\langle X, Y\rangle := \mathrm{trace}(XY)$, as in the case of $\mathcal{P}_p(\mathbf{F})$. It is easy to verify that S_n is self-dual with respect to this $\langle\ ,\ \rangle$ and that $e = \begin{pmatrix}1&0\\0&1\end{pmatrix} = (1, 1, 0, \cdots, 0) \in \mathbf{R}^n$ satisfies the condition in Fact 1. Then the calculation goes just as before, and again we have (7), (with $\varepsilon = n$). Since $\Phi_*(a\cdot\partial_y) = a\cdot\partial_t$, using Corollary 1 we get

Lemma 9. *For $a_1\cdot\partial_y, a_2\cdot\partial_y \in \mathcal{H}_0$, we have*

$$\langle a_1\cdot\partial_y, a_2\cdot\partial_y\rangle_0 = C\langle a_1, a_2\rangle,$$

where $C = \frac{1}{2}(1 + r)\varepsilon$, and $\langle\ ,\ \rangle$ is the inner product given on $\mathbf{R}^n$, where Ω lies.

Next we want to determine $g_{\alpha\bar\beta 0} = \langle\partial_{u^\alpha}, \partial_{\bar u^\beta}\rangle_0$, since by (4), (5) and (6) we then know the metric. By (3) we have to calculate the gradient of $\log\lambda$ at e. Using Fact 3 and Lemmas 2, 2′ and 4, we have, for t near e,

$$\lambda(t) = (\det\check{A}(t))^{-2}(\det\beta\check{A}(t))^{-2}\lambda(e),$$

with $(\det\check{A}(t))^2$ and $(\det\beta\check{A}(t))^2$ homogeneous functions in t of degrees l and lr respectively. Using Euler's lemma on homogeneous functions and summation convention, from

$$\log\lambda(t) = -\log(\det\check{A}(t))^2 - \log(\det\beta\check{A}(t))^2 + \log\lambda(e)$$

we get that

$$\frac{\partial\log\lambda}{\partial t^j}\cdot t^j = -\Big\{(\det\check{A}(t))^{-2}\cdot l\cdot(\det\check{A}(t))^2 + (\det\beta\check{A}(t))^{-2}\cdot lr\cdot\big(\det\beta\check{A}(t))^2\Big\} = -l(1 + r).$$

Differentiation once more gives

$$\frac{\partial^2\log\lambda}{\partial t^i\partial t^j}\cdot t^j + \frac{\partial\log\lambda}{\partial t^i} = 0.$$

By (3) we then get, with summation convention,

$$g_{\alpha\bar\beta 0} = F^i_{\alpha\bar\beta}\cdot\frac{\partial^2\log\lambda}{\partial t^i\partial t^j}\bigg|_e\cdot e^j. \tag{8}$$

By (5) we have also

$$\langle\partial_{y^i}, \partial_{y^j}\rangle_0 = 2\,\mathrm{Re}\,g_{i\bar j 0} = \frac{1}{2}\frac{\partial^2\log\lambda}{\partial t^i\partial t^j}\bigg|_e,$$

and so

$$\begin{aligned}\langle b_1\cdot\partial_u, \overline{b_2\cdot\partial_u}\rangle_0 &= F^i_{\alpha\bar\beta}b_1^\alpha\overline{b_2^\beta}\cdot\frac{\partial^2\log\lambda}{\partial t^i\partial t^j}\bigg|_e\cdot e^j = 2F^i(b_1, b_2)\langle\partial_{y^i}, \partial_{y^j}\rangle_0 e^j\\ &= 2\langle F(b_1, b_2)\cdot\partial_y, e\cdot\partial_y\rangle_0.\end{aligned}$$

By Lemma 9 this equals $2C\langle F(b_1, b_2), e\rangle = 2C\langle e, F(b_1, b_2)\rangle$. In the notation of Definition 4 of §1 we thus have

Lemma 10. *For the vectors* $b_1 \cdot \partial_u + \text{conj}, b_2 \cdot \partial_u + \text{conj} \in \mathcal{V}_0$, *we have*

$$\langle b_1 \cdot \partial_u, \overline{b_2 \cdot \partial_u} \rangle_0 = 2C\langle e, F(b_1, b_2)\rangle = 2CF_e(b_1, b_2),$$

where $C = \frac{1}{2}(1 + r)\varepsilon$.

This completes the determination of the metric, since our space $\mathfrak{D}(\Omega, F)$ is homogeneous.

3. The Bergman connection

In this section we calculate the Riemannian connection induced by the Bergman metric (the Bergman connection) on the quasi-symmetric domain $\mathfrak{D}(\Omega, F)$. Since $\mathfrak{D}(\Omega, F)$ is affinely homogeneous, and the metric is invariant under $Aff(\Omega, F)$ (and under $\text{Hol}(\Omega, F)$ too, of course), we will use the terminology of [4], to which we refer for general details.

We have

Lemma 1. *The stability subgroup of* $Aff(\Omega, F)$ *at* $o = (ie, 0)$ *is*

$$\{(A, \tilde{A}, 0, 0) \in Aff(\Omega, F) | Ae = e\} \subset Gl(\Omega, F),$$

where e *is the base point of* Ω.

Proof. Trivial.

However, it is a little bit inconvenient to work with $Aff(\Omega, F)$ since the element $\tilde{A}$ is not uniquely determined by A. (We still have the freedom of the "unitary group of F".) But since $\mathfrak{D}(\Omega, F)$ is quasi-symmetric, we have the homomorphism β: $G(\Omega)^0 \to Gl(m, \mathbf{C})$ such that $(A, \beta A) \in Gl(\Omega, F)$ for $A \in G(\Omega)^0$, where $G(\Omega)^0$ is the identity component of $G(\Omega)$. We can then consider the connected subgroup

$$G := \{(A, \beta A, a, b) | A \in G(\Omega)^0, a \in \mathbf{R}^n, b \in \mathbf{C}^m\} \tag{1}$$

of $Aff(\Omega, F)$. (See (1) of §1 for group operations.) We also write (A, a, b) for the element $(A, \beta A, a, b)$.

Lemma 2. *G is transitive on* $\mathfrak{D}(\Omega, F)$.

Proof. This follows from the fact that since $G(\Omega)$ is transitive on Ω, so is $G(\Omega)^0$, and from the fact that the subgroup $\{(a, b)\} = \{(I, a, b)\}$ of G is transitive on Φ-fibers. Recall that Φ is π-equivariant, where π: $G \ni (A, a, b) \mapsto A \in G(\Omega)^0$.

Lemma 3. *The stability subgroup of G at* $o = (ie, 0)$ *is the group*

$$K = \{(A, 0, 0) | Ae = e\} \subset G \cap Gl(\Omega, F).$$

Proof. See Lemma 1. q.e.d.

Writing $K_e = \{A \in G(\Omega)^0 | Ae = e\}$ for the stability subgroup of $G(\Omega)^0$ at e, $\mathfrak{k}_e$ for its Lie algebra and $\mathfrak{k}$ for the Lie algebra of K, from the above we have

(2) $$\mathfrak{k} = \{(X, 0, 0) | X \in \mathfrak{k}_e\} = \mathfrak{k}_e.$$

Now let $\mathfrak{g}(\Omega) = \mathfrak{k}_e + \mathfrak{p}_e$ be the Cartan decomposition of $\mathfrak{g}(\Omega)$ at e, as in Fact 1, §1, and let

(3) $$\mathfrak{m} = \{(X, a, b) | X \in \mathfrak{p}_e, a \in \mathbf{R}^n, b \in \mathbf{C}^m\}.$$

Then letting $\mathfrak{g}$ be the Lie algebra of G we have, in the terminology of [4],

Lemma 4. *$\mathfrak{D}(\Omega, F) = G/K$ is a reductive homogeneous space with respect to the decomposition*

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}.$$

Proof. That $\mathfrak{k} \cap \mathfrak{m} = \{0\}$ is clear. Since

$$[\mathfrak{k}, \mathfrak{m}] \subset \{([X, Y], a, b) | X \in \mathfrak{k}_e, Y \in \mathfrak{p}_e, a \in \mathbf{R}^n, b \in \mathbf{C}^m\}$$
$$\subset \{(Z, a, b) | Z \in \mathfrak{p}_e, a \in \mathbf{R}^n, b \in \mathbf{C}^m\} = \mathfrak{m},$$

the rest is clear. (Use the homotopy sequence for $G \to G/K$ together with the fact that $\mathfrak{D}(\Omega, F)$ is simply connected, to see that K is connected, and then we only need $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$.) q.e.d.

By [4] the Bergman connection, being G-invariant, can be expressed by a certain linear mapping $\Lambda_\mathfrak{m}$: $\mathfrak{m} \to \mathfrak{gl}(2n + 2m, \mathbf{R})$, where $2n + 2m$ is the real dimension $\mathfrak{D}(\Omega, F)$.

Now choose u_0 in the linear frame bundle of $\mathfrak{D}(\Omega, F)$, over the point o.

As in [4], it is more convenient to make the identifications

$$\mathfrak{m} = T_0(\mathfrak{D}(\Omega, F)) = \mathbf{R}^{2n+2m};$$

the first "by exp", i.e., by value of induced field at o, and the second by u_0. Then $\Lambda_\mathfrak{m}(X)$: $\mathfrak{m} \to \mathfrak{m}$ is a linear map, and we write both $\Lambda_\mathfrak{m}(X)Y$ and $\Lambda_\mathfrak{m}(X, Y)$ for its action on Y. Then using these identifications we have

(4) $$\nabla_{Y_0} X = \Lambda_\mathfrak{m}(X, Y) \quad \text{for } X, Y \in \mathfrak{m},$$

where again X is the field induced on $\mathfrak{D}(\Omega, F)$ by $X \in \mathfrak{m}$.

Now the metric gives us a symmetric bilinear form on $\mathfrak{m}$, as $\langle X, Y\rangle := \langle X, Y\rangle_0$, using the identification. There is then in [4] the following formula for the connection $\Lambda_\mathfrak{m}$ induced by the metric.

(5) $$\Lambda_\mathfrak{m}(X, Y) = \frac{1}{2}[X, Y]_\mathfrak{m} + U(X, Y) \quad \text{for } X, Y \in \mathfrak{m},$$

where U: $\mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is the symmetric, bilinear mapping defined by

(6) $$2\langle U(X, Y), Z\rangle = \langle X, [Z, Y]_\mathfrak{m}\rangle + \langle [Z, X]_\mathfrak{m}, Y\rangle$$

for $X, Y, Z \in \mathfrak{m}$, where $[X, Y]_\mathfrak{m}$ means the $\mathfrak{m}$-component of $[X, Y]$, etc.

When we apply this to our special case, we again have to write (X, a, b) instead of X, of course. Specifically, we have first (see §1)

$$\begin{aligned}(7)\qquad &[(X, a, b), (Y, c, d)]_{\mathfrak{m}} \\ &\quad = (0, Xc - Ya + 4\,\mathrm{Im}\,F(b, d), \beta(X)d - \beta(Y)b) \quad \text{for } X, Y \in \mathfrak{p}_e,\end{aligned}$$

since $[\mathfrak{p}_e, \mathfrak{p}_e] \subset \mathfrak{k}_e(\subset \mathfrak{g}(\Omega))$. Further, by (3) of §1 we get

$$(8)\qquad (X, a, b)_0 = Xe \cdot \partial_y + a \cdot \partial_x + \big(b \cdot \partial_u + \bar{b} \cdot \partial_{\bar{u}}\big).$$

We then calculate, for $(X, a, b), (Y, c, d), (Z, f, h) \in \mathfrak{m}$,

$$\begin{aligned}&\langle [(Z, f, h), (X, a, b)]_{\mathfrak{m}}, (Y, c, d)\rangle \\ &\qquad = \langle \{Za - Xf + 4\,\mathrm{Im}\,F(h, b)\} \cdot \partial_x, c \cdot \partial_x\rangle_0 \\ &\qquad\quad + \big(\langle \{\beta(Z)b - \beta(X)h\} \cdot \partial_u, \bar{d} \cdot \partial_{\bar{u}}\rangle_0 + \mathrm{conj}\big),\end{aligned}$$

where we have used the orthogonality properties in §2, Lemma 8, and also the fact that since the metric is hermitian, $\langle d_1 \cdot \partial_u, d_2 \cdot \partial_u\rangle = 0$, etc. We get then, by interchanging (X, a, b) and (Y, c, d), and adding

$$\begin{aligned}&\langle 2U(X, a, b | Y, c, d), (Z, f, h)\rangle \\ &\qquad = \langle \{Za - Xf + 4\,\mathrm{Im}\,F(h, b)\} \cdot \partial_x, c \cdot \partial_x\rangle_0 \\ &\qquad\quad + \langle \{Zc - Yf + 4\,\mathrm{Im}\,F(h, d)\} \cdot \partial_x, a \cdot \partial_x\rangle_0 \\ (9)\qquad &\qquad\quad + \big(\langle \{\beta(Z)b - \beta(X)h\} \cdot \partial_u, \bar{d} \cdot \partial_{\bar{u}}\rangle_0 + \mathrm{conj}\big) \\ &\qquad\quad + \big(\langle \{\beta(Z)d - \beta(Y)h\} \cdot \partial_u, \bar{b} \cdot \partial_{\bar{u}}\rangle_0 = \mathrm{conj}\big).\end{aligned}$$

It is more convenient now to look at cases. Then (9) tells us:

$$\text{I.}\qquad \langle 2U(X, 0, 0 | Y, 0, 0), (Z, f, h)\rangle = 0.$$

By definiteness of $\langle\ ,\ \rangle$, we get

$$U(X, 0, 0 | Y, 0, 0) = 0.$$

$$\text{II.}\qquad \langle 2U(X, 0, 0 | 0, c, 0), (Z, f, h)\rangle = -\langle Xf \cdot \partial_x, c \cdot \partial_x\rangle_0.$$

By Lemma 8 of §2 and (8) we can then write $U(X, 0, 0 | 0, c, 0) = A(X|c) \cdot \partial_x$ with

$$\langle A(X|c) \cdot \partial_x, f \cdot \partial_x\rangle_0 = -\frac{1}{2}\langle Xf \cdot \partial_x, c \cdot \partial_x\rangle_0 = -\frac{1}{2}\langle Xf \cdot \partial_y, c \cdot \partial_y\rangle_0,$$

where the last equality follows from the fact that $\langle\ ,\ \rangle$ is a hermitian metric. This is further equal to $-\frac{1}{2}C\langle Xf, c\rangle$ by §2. Now $X \in \mathfrak{p}_e$, and by Fact 1 of §1, X is symmetric with respect to the product $\langle\ ,\ \rangle$ on $\mathbf{R}^n$. So $-\frac{1}{2}C\langle Xf, c\rangle = -\frac{1}{2}C\langle f, Xc\rangle$, and by Lemma 9 of §2, this finally gives us $\langle A(X|c) \cdot \partial_x, f \cdot \partial_x\rangle_0 = -\frac{1}{2}\langle Xc \cdot \partial_x, f \cdot \partial_x\rangle_0$. Hence

$$U(X, 0, 0 | 0, c, 0) = -\tfrac{1}{2}Xc \cdot \partial_x = \big(0, -\tfrac{1}{2}Xc, 0\big).$$

Proceeding similarly in the other cases, using the information in §§1 and 2 about $\circ$, $\mathfrak{p}_e$, T: $\mathbf{R}^n \xrightarrow{\cong} \mathfrak{p}_e$, R: $\mathbf{R}^n \to \mathcal{H}(F_e)$, β: $\mathfrak{g}(\Omega) \to \mathfrak{gl}^0(m, \mathbf{C})$, and the Bergman metric, we easily find:

III. $U(X, 0, 0|0, 0, d)$,

$$= -\tfrac{1}{2}\beta(X)d \cdot \partial_u - \tfrac{1}{2}\overline{\beta(X)d} \cdot \partial_{\bar{u}} = (0, 0, -\tfrac{1}{2}\beta(X)d).$$

IV. $U(0, a, 0|0, c, 0) = (a \circ c) \cdot \partial_y = (T_{a \circ c}, 0, 0)$.

V. $U(0, a, 0|0, d) = iR_a d \cdot \partial_u + \text{conj} = (0, 0, iR_a d)$.

VI. $U(0, 0, b|0, 0, d) = 2 \operatorname{Re} F(b, d) \cdot \partial_y = (T_{2 \operatorname{Re} F(b,d)}, 0, 0)$.

Using I, $\cdots$, VI we express all terms in the expansion of $\Lambda_{\mathfrak{m}}(X, a, b)(Y, c, d)$ arising from the symmetric mapping U, put these terms and (7) into formula (5), and obtain

Proposition 1. *With respect to the decomposition* $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ *in Lemma 4 for the indecomposable, quasi-symmetric domain* $\mathfrak{D}(\Omega, F) = G/K$, *the Bergman connection is given by*

$$\Lambda_{\mathfrak{m}}(X, a, b)(Y, c, d)$$
$$= \left(T_{a \circ c + 2 \operatorname{Re} F(b,d)}, -Ya + 2 \operatorname{Im} F(b, d), -\beta(Y)b + \sqrt{-1}\,(R_a d + R_c b)\right).$$

In order to simplify the appearance and handling of this formula, we introduce "a more complex notation". For already the component b in (X, a, b) stand for (at o) the vector $b \cdot \partial_u + \bar{b} \cdot \partial_{\bar{u}}$, while X stands for $Xe \cdot \partial_y$ and a stands for $a \cdot \partial_x$. Since $\mathfrak{p}_e \ni X \mapsto Xe \in T_0(\mathfrak{D}(\Omega, F))$ is a linear isomorphism, we can write Xe instead of X, and further, we write $a = a' + ia''$ for $a' \cdot \partial_x + a'' \cdot \partial_y = a \cdot \partial_z + \bar{a} \cdot \partial_{\bar{z}}$ at o, with $a', a'' \in \mathbf{R}^n$, just as we write b for $b \cdot \partial_u + \bar{b} \cdot \partial_{\bar{u}}$ at o. Denoting by $\mathfrak{m}_{\mathbf{C}}$ the space $\mathbf{C}^n \times \mathbf{C}^m$, we have therefore an isomorphism

$$\mathfrak{m}_{\mathbf{C}} \ni (a, b) \mapsto a \cdot \partial_z + \bar{a} \cdot \partial_{\bar{z}} + b \cdot \partial_u + \bar{b} \cdot \partial_{\bar{u}} \in T_o(\mathfrak{D}(\Omega, F))$$

of complex vector spaces, where of course the complex structure on T_0 is "the one given by the manifold". With the isomorphism $\mathfrak{m} \ni (X, a', b) \mapsto (a' + iXe, b) \in \mathfrak{m}_{\mathbf{C}}$ with inverse $(a' + ia'', b) \to (T_{a''}, a', b)$, the identifications between $\mathfrak{m}$, $\mathfrak{m}_{\mathbf{C}}$ and $T_0(\mathfrak{D}(\Omega, F))$ are compatible.

When we talk about the *field* generated on $\mathfrak{D}(\Omega, F)$ by $(a, b) \in \mathfrak{m}_{\mathbf{C}}$, we mean of course, as before, the field generated by $(T_{a''}, a', b)$, which agrees with the field $a \cdot \partial_z + \bar{a} \cdot \partial_{\bar{z}} + b \cdot \partial_u + \bar{b} \cdot \partial_{\bar{u}}$ only at the origin o, in general. For simplicity we continue to write $\Lambda_{\mathfrak{m}}$. Then we translate

$$\Lambda_{\mathfrak{m}}(a' + ia'', b)(c' + ic'', d)$$
$$= \Lambda_{\mathfrak{m}}(T_{a''}, a', b)(T_{c''}, c', d)$$
$$= i(a' \circ (c' + ic'')) + 2F(d, b), R_{a'}d + R_{c' + ic''}b).$$

Here we have used the definition of T, the fact that $\operatorname{Im} F(b, d) = -\operatorname{Im} F(d, b)$, the commutativity of $\circ$, and we have extended $\circ$ bilinearly to a product $\circ : \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{C}^n$. We have also used

$$\beta(T_a) = R_a \text{ for } a \in \mathbf{R}^n, \tag{10}$$

which follows easily from Fact 2 of §1.

We then have the following reformulation of $\Lambda_{\mathfrak{m}}$.

Proposition 1′. *With respect to the decomposition* $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ *in Lemma* 4 *for the indecomposable quasi-symmetric domain* $\mathfrak{D}(\Omega, F) = G/K$, *the Bergman connection is given by* $\lambda_{\mathfrak{m}}: \mathfrak{m}_{\mathbf{C}} \times \mathfrak{m}_{\mathbf{C}} \to \mathfrak{m}_{\mathbf{C}}$ *as follows*:

$$\Lambda_{\mathfrak{m}}(a, b)(c, d) = \sqrt{-1}\,(a' \circ c + 2F(d, b), R_{a'}d + R_c b),$$

where $a, c \in \mathbf{C}^n$, $b, d \in \mathbf{C}^m$ *and* $a' = \operatorname{Re} a$.

We also want to obtain an explicit expression for the covariant derivative ∇.

Proposition 2. *For the indecomposable quasi-symmetric domain* $\mathfrak{D}(\Omega, F)$ *the Bergman connection is given by*

$$\nabla_{(c\cdot\partial_z + d\cdot\partial_u)_0}(a\cdot\partial_z + b\cdot\partial_u) = \sqrt{-1}\,\{(a \circ c)\cdot\partial_z + (R_a d + R_c b)\cdot\partial_u\} \in \mathfrak{T}_0(\mathfrak{D}(\Omega, F)),$$

where $a, c \in \mathbf{C}^n$; $b, d \in \mathbf{C}^m$; $\mathfrak{T}$ *is the holomorphic tangent bundle.*

Proof. Use the ordinary and Kählerian properties of ∇, and observe ((3) of §1) that $(a, b) = (T_{a''}, a', b)$ represents the field

$$(a, b)_{(z,u)} = \{a'\cdot\partial_z + 2iF(u, b)\cdot\partial_z + b\cdot\partial_u + T_{a''}z\cdot\partial_z + R_{a''}u\cdot\partial_u\} + \text{conj}$$

(see also (10)). Then the result follows by combining (4) with Proposition 1′.

Example. The formula in Proposition 2 generalizes the expression for the Poincaré-Bergman connection on the upper half-plane $\mathcal{H}$. For here the cone is $\Omega = \{t \in \mathbf{R} | t > 0\}$ with $e = 1$, and $G(\Omega) = \{A \in \mathbf{R} | A > 0\}$ with $\mathfrak{g}(\Omega) = \mathbf{R} = \mathfrak{p}_e$, since $K_e = \{1\}$. For $a \in \mathbf{R}$ we have $T_a = a$, since $a = T_a e = T_a 1 = a \cdot 1$. Thus $a \circ c = T_a = ac$ is ordinary multiplication, and hence $\nabla_{(\partial_z)_0}\partial_z = \sqrt{-1}\,\partial_z$, which is the correct expression.

In case $\mathfrak{D}(\Omega, F)$ is symmetric, we can derive a relation between $\Lambda_{\mathfrak{m}}$ and the symmetry σ of $\mathfrak{g}(\Omega, F)$, the Lie algebra of $\operatorname{Hol}(\mathfrak{D}(\Omega, F))$. Let $\mathcal{H}$ be the stability subgroup of $\operatorname{Hol}(\mathfrak{D}(\Omega, F))$ at o, and $\mathfrak{g}(\Omega, F) = \mathfrak{h} + \mathfrak{p}$ the Cartan decomposition at o. It is clear that we have

$$\mathfrak{p} = \frac{1 - \sigma}{2}\mathfrak{m}, \tag{11}$$

since any vector $X \in \mathfrak{m}$ decomposes as $X = \frac{1}{2}(1 + \sigma)X + \frac{1}{2}(1 - \sigma)X \in \mathfrak{h} + \mathfrak{p}$, and vectors in $\mathfrak{h}$ do not give any tangent vectors at o. Since we have to obtain *all* tangent vectors, (11) must hold. This also follows from [8], where it is stated that in the decomposition $\mathfrak{g}(\Omega, F) = \mathfrak{g}_{-1} + \mathfrak{g}_{-1/2} + \mathfrak{g}_0 + \mathfrak{g}_{1/2} + \mathfrak{g}_1$ in §1, the involution σ reverses gradation, i.e., $\sigma(\mathfrak{g}_\nu) = \mathfrak{g}_{-\nu}$. So if $Z \in \mathfrak{p}$, then with $Z_{-\nu} \in \mathfrak{g}_{-\nu}$, $\nu = 0, 1/2, 1$, and $\sigma Z_0 = -Z_0$ we can write $Z = Z_{-1} + Z_{-1/2} + 2Z_0 - \sigma Z_{-1/2} - \sigma Z_{-1}$. But $Z_{-1} + Z_{-1/2} + Z_0 \in \mathfrak{m}$, and $Z = (Z_{-1} + Z_{-1/2} + Z_0) - \sigma(Z_{-1} + Z_{-1/2} + Z_0)$. We now have

Proposition 3. *If the indecomposable quasi-symmetric domain* $\mathfrak{D}(\Omega, F)$ *is symmetric, then the above* $\Lambda_\mathfrak{m}$ *satisfies*

$$\frac{1-\sigma}{2}\Lambda_\mathfrak{m}(X, Y) = \left[\frac{1+\sigma}{2}X, \frac{1-\sigma}{2}Y\right] \quad \text{for } X, Y \in \mathfrak{m},$$

where σ *is the involution on the Lie algebra of* $\mathrm{Hol}(\mathfrak{D}(\Omega, F))$.

Proof. Let $X, Y \in \mathfrak{m}$, and $\check{X} := \frac{1}{2}(1-\sigma)X$, $\check{Y} := \frac{1}{2}(1-\sigma)Y \in \mathfrak{p}$. Then $\Lambda_\mathfrak{m}(X)Y = \nabla_{Y_0}X$. We can also express ∇ by a $\Lambda_\mathfrak{p}$ with respect to the decomposition $\mathfrak{g}(\Omega, F) = \mathfrak{h} + \mathfrak{p}$, but this $\Lambda_\mathfrak{p}$ is zero in the symmetric case. (The Bergman metric induces the canonical connection on a symmetric space, [4], and this is given by $\Lambda_\mathfrak{p} = 0$.) So

$$\begin{aligned} 0 &= \left(\Lambda_\mathfrak{p}(\check{X})\check{Y}\right)_0 = \nabla_{\check{Y}_0}\check{X} = \nabla_{Y_0}\check{X} = \nabla_{\check{X}_0}Y + [Y, \check{X}]_0 = \nabla_{X_0}Y + [Y, \check{X}]_0 \\ &= \nabla_{Y_0}X + [X, Y]_0 + [Y, \check{X}]_0 = (\Lambda_\mathfrak{m}(X)Y)_0 + [X - \check{X}, Y]_0 \\ &= (\Lambda_\mathfrak{m}(X)Y)_0 + \left[\frac{1+\sigma}{2}X, Y\right]_0, \end{aligned}$$

where the brackets are field brackets. Now the mapping $\mathfrak{g} \to \{\text{vector fields}\}$ is an antihomomorphism, and so

$$(\Lambda_\mathfrak{m}(X)Y)_0 = \left[\frac{1+\sigma}{2}X, Y\right]_0,$$

where the bracket now is an algebra bracket. Since these tangent vectors are equal, so are the $\mathfrak{p}$-components of the indicated algebra elements, i.e.,

$$\frac{1-\sigma}{2}\Lambda_\mathfrak{m}(X, Y) = \frac{1-\sigma}{2}\left[\frac{1+\sigma}{2}X, Y\right] = \left[\frac{1+\sigma}{2}X, \frac{1-\sigma}{2}Y\right].$$

4. Curvature

In [4] there is the following formula for the Riemannian curvature:

$$R(X, Y)_0 = [\Lambda_\mathfrak{m}(X), \Lambda_\mathfrak{m}(Y)] - \Lambda_\mathfrak{m}([X, Y]_\mathfrak{m}) - \lambda([X, Y]_\mathfrak{k})$$

as a mapping from $\mathfrak{m}$ to $\mathfrak{m}$, where $X, Y \in \mathfrak{m}$ and λ is, in this section, the linear isotropy representation (and $[\ ,\ ]_\mathfrak{m}$ and $[\ ,\ ]_\mathfrak{k}$ mean $\mathfrak{m}$- and $\mathfrak{k}$-components of brackets, of course). One checks easily that via the identification $\mathfrak{m} = T_0(\mathfrak{D}(\Omega, F))$, the linear isotropy representation is $A \mapsto (\operatorname{ad} A)|_\mathfrak{m}$ for $A = (A, 0, 0) \in K$. We use our $\mathfrak{m}_C$ instead, and recall the identification $\mathfrak{m} = \mathfrak{m}_C$ of

§3: $\mathfrak{m} \ni (X, a', b) \mapsto (a' + iXe, b) \in \mathfrak{m}_{\mathbf{C}}$ with inverse $(a' + ia'', b) \mapsto (T_{a''}, a', b)$. Now we calculate the three terms separately in the curvature expression:

$$R(a, b|c, d)(f, h) = [\Lambda_{\mathfrak{m}}(a, b), \Lambda_{\mathfrak{m}}(c, d)](f, h) - \Lambda_{\mathfrak{m}}([(a, b), (c, d)]_{\mathfrak{m}_{\mathbf{C}}})(f, h) - \lambda([(a, b), (c, d)]_{\mathfrak{k}})(f, h),$$

where $a, c, f \in \mathbf{C}^n$, $b, d, h \in \mathbf{C}^m$. First we get

$$(1) \qquad [(a, b), (c, d)]_{\mathfrak{m}_{\mathbf{C}}} = [(T_{a''}, a', b), (T_{c''}, c', d)]_{\mathfrak{m}} = (T_{a''}c' - T_{c''}a' + 4 \operatorname{Im} F(b, d), R_{a''}d - R_{c''}b),$$

and similarly

$$(2) \qquad [(a, b), (c, d)]_{\mathfrak{k}} = ([T_{a''}, T_{c''}], 0, 0).$$

By a straightforward calculation, we now find

$$(3) \qquad \begin{aligned} &[\Lambda_{\mathfrak{m}}(a, b), \Lambda_{\mathfrak{m}}(c, d)](f, h) \\ &\quad = ([T_{c'}, T_{a'}]f + 2F(h, R_{c'}b - R_{a'}d) + 2F(R_f b, d) - 2F(R_f d, b), \\ &\qquad [R_{c'}, R_{a'}]h - R_f R_{c'}b + R_f R_{a'}d + R_{2F(h,b)}d - R_{2F(h,d)}b). \end{aligned}$$

$$(4) \qquad \begin{aligned} &-\Lambda_{\mathfrak{m}}([(a, b), (c, d)]_{\mathfrak{m}_{\mathbf{C}}})(f, h) \\ &\quad = -i((a'' \circ c') \circ f - (c'' \circ a') \circ f \\ &\qquad + f \circ \operatorname{Im} F(b, d) + 2F(h, R_{a''}d - R_{c''}b), \\ &\qquad R_{a'' \circ c' - c'' \circ a' + 4 \operatorname{Im} F(b,d)}h + R_f(R_{a''}d - R_{c''}b)). \end{aligned}$$

$$(5) \qquad -\lambda([(a, b), (c, d)]_{\mathfrak{k}})(f, h) = -([T_{a''}, T_{c''}]f, [R_{a''}, R_{c''}]h).$$

Using $f \circ F(b, d) = F(R_f b, d) + F(b, R_f d)$ in (4), and then putting (3), (4) and (5) together, we obtain

Proposition 1. *For the indecomposable quasi-symmetric domain $\mathfrak{D}(\Omega, F)$ we have the curvature expression*

$$\begin{aligned} R_0(a, b|c, d)(f, h) = (&-([T_{a'}, T_{c'}] + [T_{a''}, T_{c''}])f \\ &- i(a'' \circ c' - a' \circ c'') \circ f + 2F(d, R_{\bar f}b) \\ &-2F(b, R_{\bar f}d) + 2F(h, R_{\bar c}b - R_{\bar a}d), \\ &-([R_{a'}, R_{c'}] + [R_{a''}, R_{c''}])h \\ &-iR_{a'' \circ c' - a' \circ c''}h - iR_{4 \operatorname{Im} F(b,d)}h + R_{2F(h,b)}d \\ &-R_{2F(h,d)}b - R_f(R_{\bar c}b - R_{\bar a}d)) \in \mathfrak{m}_{\mathbf{C}}, \end{aligned}$$

where $(a, b) \in \mathfrak{m}_{\mathbf{C}} = \mathbf{C}^n \times \mathbf{C}^m$ with $a' = \operatorname{Re} a$, $a'' = \operatorname{Im} a$, etc.

Also for the curvature do we want to obtain a more direct expression, in terms of ∂_z, ∂_u, etc. The calculation here is quite straightforward, but somewhat lengthy. The main point is to use

$$(a, b)_0 = a' \cdot \partial_x + a'' \cdot \partial_y + b \cdot \partial_u + \bar{b} \cdot \partial_{\bar{u}} = a \cdot \partial_z + b \cdot \partial_u + \text{conj},$$

and the Kähler conditions on the curvature, namely: Only
$R(a \cdot \partial_z + b \cdot \partial_u | \overline{c \cdot \partial_z + d \cdot \partial_u})$ and $R(\overline{a \cdot \partial_z + b \cdot \partial_u} | c \cdot \partial_z + d \cdot \partial_u)$
can be different from zero, and each sends (1.0)-vectors to (1, 0)-vectors, and (0, 1)-vectors to (0, 1)-vectors. We find

Proposition 2. *For the indecomposable quasi-symmetric domain* $\mathfrak{D}(\Omega, F)$ *we have*

$$\begin{aligned} R_0(a \cdot \partial_z + b \cdot \partial_u | \overline{c \cdot \partial_z + d \cdot \partial_u})(f \cdot \partial_z + h \cdot \partial_u) \\ = -\left\{\tfrac{1}{2}[T_a, T_{\bar{c}}]f + \tfrac{1}{2}(a \circ \bar{c}) \circ f + 2F(b, R_{\bar{f}} d) + 2F(h, R_{\bar{a}} d)\right\} \cdot \partial_z \\ - \left\{R_a R_{\bar{c}} h + R_f R_{\bar{c}} b + R_{2F(b,d)} h + R_{2F(h,d)} b\right\} \cdot \partial_u, \end{aligned}$$

where $a, c, f \in \mathbf{C}^n$, $b, d, h \in \mathbf{C}^m$, *and also* T *has been extended linearly to a map* $\mathbf{C}^n \to \mathfrak{gl}(n, \mathbf{C})$.

Example. For the upper half-plane again, we get (see the example in §3)

$$R_0(\partial_z | \overline{\partial_z})\partial_z = -\tfrac{1}{2}\partial_z,$$

where the origin o is the point i. This is of course the well-known expression for the curvature.

Finally we calculate the holomorphic sectional curvature, or, being no more complicated, the bisectional curvature.

For two vectors Z, W of type (1, 0) at o with $\langle Z, \bar{Z} \rangle_0 = \langle W, \bar{W} \rangle_0 = 1$, the bisectional curvature determined by the complex lines Z and W is

$$K(Z, W) = \langle R(Z, \bar{Z})W, \bar{W} \rangle_0 = K(W, Z) \in \mathbf{R}.$$

Using Lemmas 8, 9, 10 of §2 and Proposition 2 we calculate

$$\begin{aligned} \langle R(a \cdot \partial_z + b \cdot \partial_u | \overline{a \cdot \partial_z + b \cdot \partial_u})(f \cdot \partial_z + h \cdot \partial_u), \overline{f \cdot \partial_z + h \cdot \partial_u} \rangle_0 \\ = -\left\langle \left\{\tfrac{1}{2}[T_a, T_{\bar{a}}]f + \tfrac{1}{2}(a \circ \bar{a}) \circ f + 2F(b, R_{\bar{f}} b)\right) \right. \\ \left. + 2F(h, R_{\bar{a}} b)\right\} \cdot \partial_z, \overline{f \cdot \partial_z} \rangle_0 \\ - \left\langle \left\{R_a R_{\bar{a}} h + R_f R_{\bar{a}} b + R_{2F(b,b)} h + R_{2F(h,b)} b\right\} \cdot \partial_u, \overline{h \cdot \partial_u} \right\rangle_0. \end{aligned} \tag{6}$$

We get

$$\begin{aligned} \langle a_1 \cdot \partial_z, a_2 \cdot \partial_{\bar{z}} \rangle_0 &= \tfrac{1}{4}\{\langle a_1 \cdot \partial_x, a_2 \cdot \partial_x \rangle_0 + \langle a_1 \cdot \partial_y, a_2 \cdot \partial_y \rangle_0\} \\ &= \tfrac{1}{2} C\langle a_1, a_2 \rangle \end{aligned}$$

by using Lemma 8, 9. Also

$$\langle a \circ (\bar{a} \circ f), \bar{f} \rangle = \langle T_a(\bar{a} \circ f), \bar{f} \rangle = \langle \bar{a} \circ f, T'_a \bar{f} \rangle = \langle \bar{a} \circ f, a \circ \bar{f} \rangle,$$

etc., and

$$\langle v, F(u_1, u_2) \rangle = 2\langle e, F(R_v u_1, u_2) \rangle = 2\langle e, F(u_1, R_{\bar{v}} u_2) \rangle$$

for $v \in \mathbf{C}^n$, $u_1, u_2 \in \mathbf{C}^m$. Thus (6) implies, in consequence of Lemma 10,

Proposition 3. *For the indecomposable quasi-symmetric domain* $\mathfrak{D}(\Omega, F)$ *the holomorphic bisectional curvature determined by the vectors* $Z = a \cdot \partial_z + b \cdot \partial_u$, $W = f \cdot \partial_z + h \cdot \partial_u$ *at* o *with* $\langle Z, \bar{Z} \rangle_0 = \langle W, \bar{W} \rangle_0 = 1$ *is*

$$\begin{aligned} K(Z, W) = -C\Big\{ &\tfrac{1}{4}\big[\langle a \circ \bar{f}, \bar{a} \circ f \rangle + \langle a \circ \bar{a}, f \circ \bar{f} \rangle - \langle a \circ f, \bar{a} \circ \bar{f} \rangle\big] \\ &+ 4 \operatorname{Re}\langle e, F(R_{\bar{a}} b, R_{\bar{f}} h) \rangle \\ &+ 2\langle e, F(R_{\bar{a}} h, R_{\bar{a}} h) \rangle + 2\langle e, F(R_{\bar{f}} b, R_{\bar{f}} b) \rangle \\ &+ 2\langle F(b, b), F(h, h) \rangle + 2\langle F(b, h), F(h, b) \rangle \Big\}, \end{aligned}$$

where C *is the constant in Lemmas* 9, 10 *of* §2.

In particular, we get

Corollary 1. *For the indecomposable quasi-symmetric domain* $\mathfrak{D}(\Omega, F)$ *the holomorphic sectional curvature determined by the vector* $Z = a \cdot \partial_z + b \cdot \partial_u$ *at* o *with* $\langle Z, \bar{Z} \rangle_0 = 1$ *is*

$$\begin{aligned} K(Z) = -C\Big\{ &\tfrac{1}{4}\big[2\langle a \circ \bar{a}, a \circ \bar{a} \rangle - \langle a \circ a, \bar{a} \circ \bar{a} \rangle\big] \\ &+ 8\langle e, F(R_{\bar{a}} b, R_{\bar{a}} b) \rangle \\ &+ 4\langle F(b, b), F(b, b) \rangle \Big\} \leqslant 0. \end{aligned}$$

Proof. We only have to prove the last statement. Inside { } the last term is positive for $b \neq 0$ since $F(b, b) \in \bar{\Omega} \subset \mathbf{R}^n$, and the middle term is nonnegative since $e, F(R_{\bar{a}} b, R_{\bar{a}} b) \in \bar{\Omega}$ and the cone is self-dual. To calculate the first term, we put here $a = \alpha + i\beta$ with α, β real (rather than a', a'', in order to avoid too many primes), and $\alpha\beta = \beta\alpha$ instead of $\alpha \circ \beta = \beta \circ \alpha$, etc. Then

$$\begin{aligned} (\langle \alpha^2 + \beta^2, \alpha^2 + \beta^2 \rangle - \langle \alpha^2 - \beta^2, \alpha^2 - \beta^2 \rangle) \\ + (\langle \alpha^2 + \beta^2, \alpha^2 + \beta^2 \rangle - \langle 2\alpha\beta, 2\alpha\beta \rangle) \\ = 4\langle \alpha^2, \beta^2 \rangle + \langle (\alpha + \beta)^2, (\alpha - \beta)^2 \rangle. \end{aligned}$$

Now by [8], $\alpha^2 \in \bar{\Omega}$ for any $\alpha \in \mathbf{R}^n$, so here $\alpha^2, \beta^2, (\alpha + \beta)^2, (\alpha - \beta)^2 \in \bar{\Omega}$. Since Ω is self-dual, the inner products between these elements are nonnegative, and the corollary follows from the fact that $C > 0$.

5. Symmetric domains

In this section we shall find necessary and sufficient conditions for an indecomposable quasi-symmetric domain to by symmetric. A Riemannian manifold is Riemannian locally symmetric if and only if $\nabla R = 0$, and a complete simply connected Riemannian locally symmetric space is Riemannian symmetric [4]. Now a homogeneous Siegel domain is complete and contractible, and hence is symmetric if and only if $\nabla R = 0$. So we have to calculate ∇R. For this purpose, it is practical to consider cases, i.e., we calculate $(\nabla_W R)(X, Y)$ where $W, X, Y \in T_\circ(\mathfrak{D}(\Omega, F))$ have components either "along ∂_z" or "along ∂_u".

Now $(a, b) \in \mathfrak{m}_C$ induces a field with value $a \cdot \partial_z + b \cdot \partial_u$ + conj at o, and we write $\{a, b\}$ for the vector $a \cdot \partial_z + b \cdot \partial_u$ at o. By using the second Bianchi identity and the Kählerian properties of the curvature, it is then sufficient to calculate $(\nabla_W R)(X, \bar{Y})Z$ with the vectors W, X, Y and Z as in the following table:

	1	2	3	4	5	6	7	8	9	10	11	12
W	$(s, 0)$	$(s, 0)$	$(s, 0)$	$(s, 0)$	$(s, 0)$	$(s, 0)$	$(0, w)$	$(0, w)$	$(0, w)$	$(0, w)$	$(0, w)$	$(0, w)$
X	$\{a, 0\}$	$\{a, 0\}$	$\{a, 0\}$	$\{a, 0\}$	$\{a, 0\}$	$\{a, 0\}$	$\{a, 0\}$	$\{a, 0\}$	$\{a, 0\}$	$\{a, 0\}$	$\{0, b\}$	$\{0, b\}$
Y	$\{c, 0\}$	$\{c, 0\}$	$\{0, d\}$	$\{0, d\}$	$\{0, d\}$	$\{0, d\}$	$\{0, d\}$	$\{0, d\}$	$\{0, d\}$	$\{0, d\}$	$\{0, d\}$	$\{0, d\}$
Z	$\{f, 0\}$	$\{0, h\}$	$\{f, 0\}$	$\{\overline{f, 0}\}$	$\{0, h\}$	$\{\overline{0, h}\}$	$\{f, 0\}$	$\{\overline{f, 0}\}$	$\{0, h\}$	$\{\overline{0, h}\}$	$\{f, 0\}$	$\{0, h\}$

We use the following formula for ∇R:

$$(\nabla_{W_0} R)(X, Y)Z = \nabla_{R_0(X, Y)Z} W - R_0(X, Y)\nabla_Z W - R_0(\nabla_X W, Y)Z - R_0(X, \nabla_Y W)Z,$$

where $X, Y, Z \in T_0(\mathfrak{D}(\Omega, F))$, and W is the vector field induced by $W \in \mathfrak{g}$. This formula follows from the one given in the Appendix by observing that $\Lambda_{\mathfrak{m}}(W) = \nabla_{X_0}$ for zero torsion and $W \in \mathfrak{m}$, and that $\nabla_{X_0} W = \Lambda_{\mathfrak{m}}(W)X = 0$ for $W \in \mathfrak{k}$. The reason why this is a more convenient expression than

$$(\nabla_W R)(X, Y)Z = \nabla_W\{R(X, Y)Z\} - R(X, Y)\nabla_W Z - R(\nabla_W X, Y)Z - R(X, \nabla_W Y)Z,$$

is that in the latter we would have to know how $R(X, Y)Z$ varies from point to point, and also to differentiate this field. To help the reader check the calculations, we collect here the necessary formulas, where X, Y, Z, W are as in the above table. (Observe that W is a real field, and hence $\nabla_{\bar{Y}} W = \overline{\nabla_Y W}$.) The first such formula is

$$(1) \qquad (\nabla_{W_0} R)(X, \bar{Y})Z = \nabla_{R_0(X, \bar{Y})Z} W - R_0(X, \bar{Y})\nabla_Z W - R_0(\nabla_X W, \bar{Y})Z - R_0(X, \overline{\nabla_Y W})Z.$$

Also writing $R\{a, b|\overline{c, d}\}\{f, h\}$ for $R(a \cdot \partial_z + b \cdot \partial_u|\overline{c \cdot \partial_z + d \cdot \partial_u})(f \cdot \partial_z + h \cdot \partial_u)$, we have (Proposition 2 of §4)

$$R_0\{a, b|\overline{c, d}\}\{f, h\}$$
$$= -\{\tfrac{1}{2}[T_a, T_{\bar c}]f + \tfrac{1}{2}(a \circ \bar c) \circ f + 2F(b, R_{\bar f}d) + 2F(h, R_{\bar a}d), \quad (2)$$
$$R_a R_{\bar c} h + R_f R_{\bar c} b + R_{2F(b,d)}h + R_{2F(h,d)}b\}.$$

Further

$$\nabla_{\{a,b\}}(s, w) = \nabla_{\{a,b\}}(\{s, w\} + \{\overline{s, w}\}) = \nabla_{\{a,b\}}\{s, w\}$$
$$= (1, 0)\text{-component of } \nabla_{(a,b)}(s, w).$$

Since $\Lambda_m(s, w)(a, b) = \nabla_{(a,b)}(s, w)$, by Proposition 1′ of §3 we then have

$$\nabla_{\{a,b\}}(s, w) = \sqrt{-1}\ \{a \circ s' + 2F(b, w), R_a w + R_{s'} b\}, \quad (3)$$

$$\nabla_{\{\overline{a,b}\}}(s, w) = \overline{\nabla_{\{a,b\}}(s, w)}. \quad (4)$$

We also use

$$a \circ F(u, v) = F(R_a u, v) + F(u, R_{\bar a} v), \quad (5)$$

$$R_{a_1 \circ a_2} = R_{a_1} R_{a_2} + R_{a_2} R_{a_1}. \quad (6)$$

Finally, it is convenient to note that from (2) we have

$$R\{a, 0|\overline{0, d}\}\{f, 0\} = 0, \quad R\{0, b|\overline{c, 0}\}\{0, h\} = 0, \quad (7)$$

and that by the Kählerian properties of the curvature we have

$$R\{a, b|\overline{c, d}\}\{\overline{f, h}\} = -\overline{R\{c, d|\overline{a, b}\}\{f, h\}}. \quad (8)$$

In the table we do not have to calculate Case 1. For then all vectors are "∂_z-like", and all formulas used will be those which we have if there is no F, i.e., if we are dealing with the tube domain $\mathfrak{D}(\Omega)$: Now $\mathfrak{D}(\Omega)$ is symmetric, and hence $\nabla R = 0$ in that case. For the other cases we have to apply the method of brutal force, but the calculation is quite straightforward. The result is that $(\nabla_W R)(X, Y)Z \equiv 0$ in all but the last four cases. We find:

Case 9.

$$-2i\{0, [R_a R_{F(h,d)} - R_{F(R_a h,d)}]w + [R_a R_{F(w,d)} - R_{F(R_a w,d)}]h\}.$$

Case 12.

$$-4i\{(F(R_{F(b,d)}h, w) - F(b, R_{F(w,h)}d))$$
$$+ (F(R_{F(h,d)}b, w) - F(h, R_{F(w,b)}d)), 0\}.$$

In the final stage of the calculation of Case 9, we used the following identity:

$$R_{F(u,v)}R_a - R_{F(u,R_{\bar a}v)} = -[R_a R_{F(u,v)} - R_{F(R_a u,v)}] \quad (9)$$

for $a \in \mathbf{C}^n$, $u, v \in \mathbf{C}^m$, which we prove as follows:

$$R_a R_{F(u,v)} + R_{F(u,v)} R_a = R_{a \circ F(u,v)} = R_{F(R_a u, v) + F(u, R_{\bar{a}} v)}$$
$$= R_{F(R_a u, v)} + R_{F(u, R_{\bar{a}} v)}.$$

Consider Case 9. The expression there is symmetric in w and h, and hence is identically zero if and only if

$$(10) \qquad R_a R_{F(h,d)} h \equiv R_{F(R_a h, d)} h.$$

Cases 10 and 11 give the same kind of condition.

In Case 12, the expression is symmetric in b and h, and hence is identically zero if and only if

$$(11) \qquad F(R_{F(b,d)} b, w) \equiv F(b, R_{F(w,b)} d).$$

We claim

Lemma 1. *Conditions* (10) *and* (11) *are equivalent.*

Proof. First recall that $\frac{1}{2}\langle a, F(u, v)\rangle = \langle e, F(R_a u, v)\rangle = \langle e, F(u, R_{\bar{a}} v)\rangle$ for $a \in \mathbf{C}^n$, $u, v \in \mathbf{C}^m$, where $\langle\ ,\ \rangle$ is the $\mathbf{C}$-bilinear extension of the inner product on $\mathbf{R}^n$. Now assume (10) holds. Then

$$\frac{1}{2}\langle a, F(R_{F(b,d)} b, w)\rangle = \langle e, F(R_a R_{F(b,d)} b, w)\rangle$$
$$= \langle e, F(R_{F(R_a b, d)} b, w)\rangle = \frac{1}{2}\langle F(R_a b, d), F(b, w)\rangle$$
$$= \langle e, R(R_a b, R_{\overline{F(b,w)}} d)\rangle = \frac{1}{2}\langle a, F(b, R_{F(w,b)} d)\rangle$$

for all a, b, d, w. Hence (11) holds.

Since $F_e(u, v) = \langle e, F(u, v)\rangle$ is definite, the converse calculation also works, showing that (11) implies (10). q.e.d.

Without loss of generality we can restrict a in (10) to be in $\mathbf{R}^n$ and get

Theorem 1. *An indecomposable quasi-symmetric domain* $\mathfrak{D}(\Omega, F)$ *is symmetric if and only if the following equivalent conditions hold*:

(i) $R_a R_{F(b,d)} b = R_{F(R_a b, d)} b, \quad \forall a \in \mathbf{R}^n, \forall b, d \in \mathbf{C}^m,$

(ii) $F(R_{F(b,d)} b, w) = F(b, R_{F(w,b)} d), \quad \forall b, d, w \in \mathbf{C}^m.$

Remark. This theorem was proved algebraically by Satake (with condition (ii). Observe that his F is conjugate to our F. See [8]). His statement is somewhat stronger, since he does not suppose that $\mathfrak{D}(\Omega, f)$ is indecomposable and quasi-symmetric. He states that symmetry $\Leftrightarrow$ quasi-symmetry + condition (ii).

Appendix

We prove the formula for ∇R, or more generally, for $\nabla \alpha$, where α is any G-invariant tensor on G/K.

Proposition. *Let $M = G/K$ be a reductive homogeneous space with respect to the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ of the Lie algebra of G, $\mathfrak{k}$ being the Lie algebra of K, and let $\Lambda_{\mathfrak{m}}: \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ be an invariant connection on M. If α is any G-invariant tensor on M of type (r, s), then*

$$(\nabla_{W_0}\alpha)(X_1, \cdots, X_s) = \Lambda_{\mathfrak{m}}(W)\alpha(X_1, \cdots, X_s) - \sum_{j=1}^{s} \alpha(X_1, \cdots, \Lambda_{\mathfrak{m}}(W)X_j, \cdots, X_s) \in \mathfrak{m}^{\otimes r},$$

where o is the origin K of M, $W, X_1, \cdots, X_s \in \mathfrak{m} \cong T_0M$, $\alpha(X_1, \cdots, X_s) \in \mathfrak{m}^{\otimes s}$, and $\Lambda_{\mathfrak{m}}(W) \in \mathrm{End}(\mathfrak{m}^{\otimes s})$ is defined as $\Lambda_{\mathfrak{m}}(W) \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id} + \cdots + \mathrm{id} \otimes \cdots \otimes \mathrm{id} \otimes \Lambda_{\mathfrak{m}}(W)$ for $s > 0$ and as zero for $s = 0$.

Proof. We prove this in the case $r = 1$, the more general case being just notationally more complicated. Let $W, X_1, \cdots, X_s$ also denote the fields generated by these elements of $\mathfrak{m}$. Then

$$(\nabla_{W_0}\alpha)(X_1, \cdots, X_s) = \nabla_{W_0}\{\alpha(X_1, \cdots, X_s)\} - \sum_{j=1}^{s} \alpha_0(X_1, \cdots, \nabla_W X_j, \cdots, X_s).$$

Now

$$\nabla_{W_0}X_j = \nabla_{X_{j0}}W + T_0(W, X_j) + [W, X_j]_0 = \Lambda_{\mathfrak{m}}(W)X_j + [W, X_j]_0,$$

where T is the torsion [4, pp. 188–191]. Similarly,

$$\nabla_{W_0}\{\alpha(X_1, \cdots, X_s)\} = \Lambda_{\mathfrak{m}}(W)\alpha(X_1, \cdots, X_s) + [W, \alpha(X_1, \cdots, X_s)]_0.$$

So we get the formula stated *plus*

$$[W, \alpha(X_1, \cdots, X_s)]_0 - \textstyle\sum_{j=1}^{s} \alpha_0(X_1, \cdots, [W, X_j], \cdots, X_s).$$

However, if L_W is the Lie derivative, then

$$\begin{aligned}[W, \alpha(X_1, \cdots, X_s)] &= L_W\{\alpha(X_1, \cdots, X_s)\} \\ &= (L_W\alpha)(X_1, \cdots, X_s) + \sum_{j=1}^{s} \alpha(X_1, \cdots, L_W X_j, \cdots, X_s) \\ &= \sum_{j=1}^{s} \alpha(X_1, \cdots, [W, X_j], \cdots, X_s),\end{aligned}$$

by the invariance of α.

Corollary. *In the above situation, for the curvature R and the torsion T we have*

$$\text{(i)}\ (\nabla_{W_0}R)(X, Y) = [\Lambda_{\mathfrak{m}}(W), R_0(X, Y)] - R_0(\Lambda_{\mathfrak{m}}(W)X, Y) - R_0(X, \Lambda_{\mathfrak{m}}(W)Y) \in \operatorname{End}\mathfrak{m},$$

$$\text{(ii)}\ (\nabla_{W_0}T)(X, Y) = \Lambda_{\mathfrak{m}}(W)T(X, Y) - T(\Lambda_{\mathfrak{m}}(W)X, Y) - T(X, \Lambda_{\mathfrak{m}}(W)Y) \in \mathfrak{m}.$$

Proof. Applying (i) to $Z \in \mathfrak{m}$, we have a term $R_0(X, Y)\Lambda_{\mathfrak{m}}(W)Z$ in the above commutator, and this term comes from the sum in the proposition.

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